

The orthogonality of q -classical polynomials of the Hahn class: A geometrical approach

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Abstract

The idea of this review article is to discuss in a unified way the orthogonality of all positive definite polynomial solutions of the q -hypergeometric difference equation on the q -linear lattice by means of a qualitative analysis of the q -Pearson equation. Therefore, our method differs from the standard ones which are based on the Favard theorem, the three-term recurrence relation and the difference equation of hypergeometric type. Our approach enables us to extend the orthogonality relations for some well-known q -polynomials of the Hahn class to a larger set of their parameters.

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1 Introduction

The so-called q -polynomials are of great interest inside the class of special functions since they play an important role in the treatment of several problems such as Eulerian series and continued fractions [7, 13], q -algebras and quantum groups [21, 22, 32] and q -oscillators [9, 4, 16], and references therein, among others.

A q -analog of the Chebychev's discrete orthogonal polynomials is due to Markov in 1884 [6, page 43], which can be regarded as the first example of a q -polynomial family. In 1949, Hahn introduced the q -Hahn class [17] including the big q -Jacobi polynomials, on the exponential lattice although he did not use this terminology. In fact, he did not give the orthogonality relations of the big q -Jacobi polynomials in [17] which was done by Andrews and Askey [6]. During the last decades the q -polynomials have been studied by many authors from different points of view. There are two most recognized approaches. The first approach, initiated by the work of Askey and Wilson [8] (see also Andrews and Askey [6]) is based on the basic hypergeometric series [7, 15]. The second approach is due to Nikiforov and Uvarov [27, 28] and uses the analysis of difference equations on non-uniform lattices. The readers are also referred to the surveys [10, 26, 29, 31]. These approaches are associated with the so-called q -Askey scheme [19] and the Nikiforov-Uvarov scheme [29], respectively. Another approach was published in [25] where the authors proved several characterizations of the q -polynomials starting from the so-called distributional q -Pearson equation (for the non q -case see e.g. [14, 24] and references therein).

In particular, in [25] a classification of all possible families of orthogonal polynomials on the exponential lattice was established, and latter on in [5] the comparison with the q -Askey and Nikiforov-Uvarov schemes was done, resulting in two new families of orthogonal polynomials. Furthermore, an important contribution to the theory of (orthogonal) q -polynomials, and in particular, to the theory of orthogonal q -polynomials on the linear exponential lattice, appeared in the recent book [19]. The corresponding table is generally called the q -Hahn tableau (see e.g., Koornwinder [22]). The q -polynomials belonging to this class are the solutions of the q -difference equation of hypergeometric type (q -EHT) [17]

$$\sigma_1(x; q)D_{q^{-1}}D_q y(x, q) + \tau(x, q)D_q y(x, q) + \lambda(q)y(x, q) = 0. \quad (1.1)$$

One way of deriving the q -EHT (1.1) whose bounded solutions are the q -polynomials of the Hahn class, is to discretize the classical differential equation of hypergeometric type (EHT)

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \quad (1.2)$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree, respectively, and λ is a constant [1, 10, 24, 26, 28]. To this end, we can use the approximations (see e.g. [28, §13, page 142])

$$y'(x) \sim \frac{1}{1+q}[D_q y(x) + qD_{q^{-1}}y(x)] \quad \text{and} \quad y''(x) \sim \frac{2q}{1+q}D_q D_{q^{-1}}y(x) \quad \text{as} \quad q \rightarrow 1$$

for the derivatives in (1.2), where we use the standard notation for the q and q^{-1} -Jackson derivatives of $y(x)$ [15, 18], i.e.,

$$D_\zeta y(x) = \frac{y(x) - y(\zeta x)}{(1 - \zeta)x}, \quad \zeta \in \mathbb{C} \setminus \{0, \pm 1\}$$

for $x \neq 0$ and $D_\zeta y(0) = y'(0)$, provided that $y'(0)$ exists. This leads to the q -EHT (1.1) where

$$\sigma_1(x; q) := \frac{2}{1+q} \left[\sigma(x) - \frac{1}{2}(q-1)x\tau(x) \right], \quad \tau(x, q) := \tau(x), \quad \lambda(q) := \lambda, \quad y(x, q) := y(x).$$

Notice here the relations $D_q = D_{q^{-1}} + (q-1)x D_q D_{q^{-1}}$ and $D_q D_{q^{-1}} = q^{-1} D_{q^{-1}} D_q$ so that (1.1) can be rewritten in the equivalent form

$$\sigma_2(x; q)D_q D_{q^{-1}}y(x, q) + \tau(x, q)D_{q^{-1}}y(x, q) + \lambda(q)y(x, q) = 0, \quad (1.3)$$

where

$$\sigma_2(x, q) := q [\sigma_1(x, q) + (1 - q^{-1})x\tau(x, q)]. \quad (1.4)$$

It should be noted that the q -EHT (1.1) and (1.3) correspond to the second order linear difference equations of hypergeometric type on the linear exponential lattices $x(s) = c_1 q^s + c_2$ and $x(s) = c_1 q^{-s} + c_2$, respectively [1, 10, 26].

Notice also that (1.1) (or (1.3)) can be written in a very convenient form [5, 19, 20]

$$\sigma_2(x, q)D_q y(x, q) - q\sigma_1(x, q)D_{q^{-1}}y(x, q) + (q-1)x\lambda(q)y(x, q) = 0,$$

where the coefficients $\sigma_1(x; q)$ and $\sigma_2(x; q)$ are polynomials of at most 2nd degree and $\tau(x, q)$ is a 1st degree polynomial in x .

Notice that the q -EHT (1.1) can be written in the self-adjoint form

$$D_q [\sigma_1(x, q)\rho(x, q)D_{q^{-1}}y(x)] + q^{-1}\lambda(q)\rho(x, q)y(x) = 0,$$

where ρ is a function satisfying the so-called q -Pearson equation $D_q [\sigma_1(x, q)\rho] = q^{-1}\tau(x, q)\rho$ that can be written as

$$\frac{\rho(qx, q)}{\rho(x, q)} = \frac{\sigma_1(x, q) + (1 - q^{-1})x\tau(x, q)}{\sigma_1(qx, q)} = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)}, \quad (1.5)$$

or, equivalently,

$$\sigma_2(x, q)\rho(x, q) = q\sigma_1(qx, q)\rho(qx, q). \quad (1.6)$$

In this paper we study, without loss of generality, the q -EHT (1.1), assume $0 < q < 1$ and take $\lambda(q)$ as

$$\lambda(q) := \lambda_n(q) = -[n]_q [\tau'(0, q) + \frac{1}{2}[n-1]_{q^{-1}}\sigma_1''(0, q)], \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

since we are interested only in the polynomial solutions [1, 10, 26]. For more details on the q -polynomials of the q -Hahn tableau we refer the readers to the works [1, 2, 3, 5, 10, 12, 19, 22, 23, 26, 27, 28, 29, 31], and references therein.

In this paper, we deal with the orthogonality properties of the q -polynomials of the q -Hahn tableau from a different viewpoint than the one used in [19]. In [19], the authors presented a unified study of the orthogonality of q -polynomials based on the Favard Theorem. Here, the main idea is to provide a relatively simple geometrical analysis of the q -Pearson equation by taking into account every possible rational form of the polynomial coefficients of the q -difference equation. Roughly, our qualitative analysis is concerned with the examination of the behavior of the graphs of the ratio $\rho(qx, q)/\rho(x, q)$ by means of the definite right hand side (r.h.s.) of (1.5) in order to find out a suitable q -weight function. Such a qualitative analysis implies all possible orthogonality relations among the polynomial solutions of the q -difference equation in question. Moreover, it allows us to extend the orthogonality relations for some well-known q -polynomials of the Hahn class to a larger set of their parameters (see sections 4.1 and 5.1). A first attempt of using a geometrical approach for studying the orthogonality of q -polynomials of the q -Hahn class was presented in [12]. However, the study is far from being complete and only some partial results were obtained. We will fill this gap in this review paper.

Our main goal is to study each orthogonal polynomial system or sequence (OPS), which is orthogonal with respect to (w.r.t.) a q -weight function $\rho(x, q) > 0$ satisfying the q -Pearson equation as well as certain boundary conditions (BCs) to be introduced in Section 2. For each family of polynomial solutions of (1.1) we search for the ones that are orthogonal in a suitable intervals depending on the range of the parameters coming from the coefficients of (1.1) and the corresponding q -Pearson equation. Hence, in Section 2, we present the candidate intervals by inspecting the BCs as well as some preliminary results. Theorems which help to calculate q -weight functions are given in Section 3. Section 4 deals with the qualitative analysis including the theorems stating the main results of our article. The last Section concludes the paper with some final remarks.

2 The orthogonality and preliminary results

We first introduce the so-called q -Jackson integrals and afterward a well known theorem for the orthogonality of polynomial solutions of (1.1) in order to make the article self-contained [1, 11, 26].

The q -Jackson integrals for $q \in (0, 1)$ [15, 18] are defined by

$$\int_0^a f(x) d_q x = (1-q)a \sum_{j=0}^{\infty} q^j f(q^j a) \quad \text{and} \quad \int_a^0 f(x) d_q x = (1-q)(-a) \sum_{j=0}^{\infty} q^j f(q^j a) \quad (2.1)$$

if $a > 0$ and $a < 0$, respectively. Therefore, we have

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x \quad \text{and} \quad \int_a^b f(x) d_q x := \int_a^0 f(x) d_q x + \int_0^b f(x) d_q x \quad (2.2)$$

when $0 < a < b$ and $a < 0 < b$, respectively. Furthermore, we make use of the *improper* q -Jackson integrals

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j) \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{j=-\infty}^{\infty} q^j [f(q^j) + f(-q^j)] \quad (2.3)$$

where the second one is sometimes called the *bilateral q -integral*. The q^{-1} -*Jackson integrals* are defined similarly. For instance, the improper q^{-1} -*Jackson integral* on (a, ∞) is given by

$$\int_a^\infty f(x) d_{q^{-1}} x = (q^{-1} - 1)a \sum_{j=0}^\infty q^{-j} f(q^{-j}a), \quad a > 0 \quad (2.4)$$

provided that $\lim_{j \rightarrow \infty} q^{-j} f(q^{-j}a) = 0$ and the series is convergent.

Theorem 2.1 *Let ρ be a function satisfying the q -Pearson equation (1.5) in such a way that the BCs*

$$\sigma_1(x, q)\rho(x, q)x^k \Big|_{x=a, b} = \sigma_2(q^{-1}x, q)\rho(q^{-1}x, q)x^k \Big|_{x=a, b} = 0, \quad k \in \mathbb{N}_0 \quad (2.5)$$

also hold. Then the sequence $\{P_n(x, q)\}$ of polynomial solutions of (1.1) are orthogonal on (a, b) w.r.t $\rho(x, q)$ in the sense that

$$\int_a^b P_n(x, q)P_m(x, q)\rho(x, q)d_q x = d_n^2(q)\delta_{mn}, \quad (2.6)$$

where $d_n(q)$ and δ_{mn} denote the norm of the polynomials P_n and the Kronecker delta, respectively. Analogously, if the conditions

$$\sigma_2(x, q)\rho(x, q)x^k \Big|_{x=a, b} = \sigma_1(qx, q)\rho(qx, q)x^k \Big|_{x=a, b} = 0, \quad k \in \mathbb{N}_0 \quad (2.7)$$

are fulfilled, the q -polynomials then satisfy the relation

$$\int_a^b P_n(x, q)P_m(x, q)\rho(x, q)d_{q^{-1}} x = d_n^2(q)\delta_{mn}. \quad (2.8)$$

Remark 2.2 *The relation (2.6) means that the polynomials $P_n(x, q)$ are orthogonal with respect to a measure supported on the set of points $\{q^k a\}_{k \in \mathbb{N}_0}$ and $\{q^k b\}_{k \in \mathbb{N}_0}$. Since we are interested in the positive definite cases, i.e., when $\rho(x, q) > 0$, then,*

- *when $a = 0$, the measure is supported on the set of points $\{q^k b\}_{k \in \mathbb{N}_0}$ in $(0, b]$.*
- *when $a > 0$, the measure should be supported on the finite set of points $\{q^k b\}_{k=0}^N$ being $a = q^{N+1}b$.*
- *when $a < 0$, the measure is supported on the set $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^k b\}_{k \in \mathbb{N}_0}$ in $[a, 0) \cup (0, b]$.*

A similar analysis can be done for the relation (2.8).

According to Theorem 2.1, we have to determine an interval (a, b) in which ρ is q -integrable and $\rho > 0$ on the lattice points of the types $\alpha q^{\pm k}$ and $\beta q^{\pm k}$ for $k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$. Such a weight function will be a solution of the q -Pearson equation (1.5). To this end, a qualitative analysis of the q -Pearson equation is presented by a detailed inspection of the r.h.s. of (1.5). Note that the r.h.s. of (1.5) consists of the polynomial coefficients σ_1 and σ_2 of the q -EHT which can be made definite for possible forms of the coefficients. As a result, the possible behavior of ρ on the left hand side (l.h.s.) of (1.5) and the candidate intervals can be obtained accordingly.

OPSs on finite (a, b) intervals

First assume that (a, b) denotes a finite interval. We list the following possibilities for finding ρ which obeys the BCs in (2.5) or in (2.7).

PI. This is the simplest case where σ_1 vanishes at both $x = a$ and b , i.e., $\sigma_1(a, q) = \sigma_1(b, q) = 0$. Using (1.5) rewritten of the form

$$\rho(q^{-1}x, q) = \frac{q\sigma_1(x, q)}{\sigma_2(q^{-1}x, q)}\rho(x, q) \quad (2.9)$$

we see that the function $\rho(x, q)$ becomes zero at the points $q^{-k}a$ and $q^{-k}b$ for $k \in \mathbb{N}$. However, we have to take into consideration three different situations.

(i) Let $a < 0 < b$. Since the points $q^{-k}a$ and $q^{-k}b$ lie outside the interval (a, b) and BCs are fulfilled at $x = a$ and b , there could be an OPS w.r.t. a measure supported on the union of the set of points $\{aq^k\}_{k \in \mathbb{N}_0}$ and $\{bq^k\}_{k \in \mathbb{N}_0}$ in $[a, 0) \cup (0, b]$, if ρ is positive.

(ii) Let $0 < a < b$. In this case $\rho(x, q)$ vanishes at the points $q^{-k}a$ in (a, b) and $q^{-k}b$ out of (a, b) . Then, the only possibility to have an OPS on $(a, b]$ depends on the existence of N such that $q^{N+1}b = a$. This condition, however, implies that $bq^k = aq^{-(N-k)}$ and that ρ vanishes at bq^k for $k = 0, 1, \dots, N$, and, therefore, it must be rejected. The similar statement is true when $a < b < 0$, which can be obtained by a simple linear scaling transformation so that it does not represent an independent case.

(iii) Let $a = 0 < b$ (or, $a < b = 0$). This case is much more involved. First of all, if $a = 0$ is a zero of $\sigma_1(x, q)$ then it is a zero of $\sigma_2(x, q)$ as well, both containing a factor x . Therefore, the r.h.s. of q -Pearson equation (1.5) can be simplified and **PI**(i)-(ii) are not valid anymore. In fact, in this case an OPS w.r.t. a measure supported on the set of points $\{bq^k\}_{k \in \mathbb{N}_0}$ in $(0, b]$ can be defined.

PII. The relation in (1.6) suggest an alternative possibility to define an OPS on (a, b) . Namely, if $q^{-1}a$ and $q^{-1}b$ are both zeros of $\sigma_2(x, q)$, by using (1.5) rewritten of the form

$$\rho(qx, q) = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)}\rho(x, q), \quad (2.10)$$

it follows that $\rho(x, q)$ vanishes at the points q^ka and q^kb for $k \in \mathbb{N}_0$. Then two different situations appear depending on whether $a < 0 < b$ or $0 < a < b$. In the first case, $\rho(x, q) = 0$ at the points q^ka and q^kb for $k \in \mathbb{N}_0$ in $[a, b]$, which is not interesting. In the second case, the q^kb are in $[a, b]$ whereas the q^ka remain out of $[a, b]$, so that we could have an OPS if there exists N such that $q^{-N-1}a = q^{-1}b$. However, since $q^{-k}a = q^{N-k}b$, ρ vanishes at the $q^{-k}a$ which are in $[a, b]$ as well.

PIII. Let $q^{-1}a$ and b be the roots of σ_2 and σ_1 , respectively. Then we see, from (2.9) and (2.10), that $\rho = 0$ at $x = q^{-k}b$ for $k \in \mathbb{N}$ and at $x = q^ka$ for $k \in \mathbb{N}_0$. That is, if $a < 0 < b$, $\rho = 0$ on $x \in (a, 0)$ and, therefore, an OPS can not be constructed on (a, b) unless $a \rightarrow 0^-$. In this limiting case of $x \in (0, b]$, it can be possible to introduce a desired weight function supported on the set $\{bq^k\}_{k \in \mathbb{N}_0}$. If $0 < a < b$, on the other hand, ρ vanishes for $x < a$ and $x > b$. Thus there could be an OPS w.r.t. a measure supported on the finite set of points $\{q^kb\}_{k=0}^N$ provided that $q^{N+1}b = a$ for some finite N integer. Alternatively, we can define an equivalent OPS w.r.t. a measure supported on the equivalent finite set of points $\{q^{-k}a\}_{k=0}^N$ provided now that $q^{-N-1}a = b$, where N is a finite integer. Note that in the limiting case of $a \rightarrow 0^+$ the set of points $\{q^kb\}_{k \in \mathbb{N}_0}$ becomes infinity.

PIV. Assume that a and $q^{-1}b$ are the roots of $\sigma_1(x, q)$ and $\sigma_2(x, q)$, respectively. Then, from (2.9) and (2.10), it follows that $\rho(x, q)$ vanishes at the points $q^{-k}a$, $k \in \mathbb{N}$ and q^kb , $k \in \mathbb{N}_0$. So, if $a < 0 < b$, it is not possible to find a weight function satisfying the BCs. Nevertheless, as in **PIII**, in the limiting case of $b \rightarrow 0^+$ an OPS w.r.t. a measure supported at the points q^ka , $k \in \mathbb{N}_0$ in $[a, 0)$ can be constructed. If $0 < a < b$, there is no possibility to introduce an OPS. Note that when $a = 0 < b$, an OPS also does not exist.

OPSs on infinite intervals

Assume now that (a, b) is an infinite interval. Without any loss of generality, let a be a finite number and $b \rightarrow \infty$. In fact, the system on the infinite interval $(-\infty, b)$ is not independent which may be transformed into (a, ∞) on replacing x by $-x$. Obviously one BC in (2.5) reads as

$$\lim_{b \rightarrow \infty} \sigma_1(b, q) \rho(b, q) b^k = 0 \quad \text{or} \quad \lim_{b \rightarrow \infty} \sigma_2(b, q) \rho(b, q) b^k = 0, \quad k \in \mathbb{N}_0,$$

and there are additional cases for $x = a$.

PV. If $x = a \neq 0$ is root of $\sigma_1(x, q)$ then, from (2.9), $\rho(x, q)$ vanishes at the points $q^{-k}a$ for $k \in \mathbb{N}$ which are interior points of (a, ∞) when $a > 0$. Therefore there is no OPS on (a, ∞) for $a > 0$. If $a < 0$ we can find a q -weight function in $[a, 0) \cup (0, \infty)$ supported on the union of the sets $\{q^k a\}_{k \in \mathbb{N}_0}$ and $\{q^{\pm k} \alpha\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ where α can be taken as unity. If $a = 0$, on the other hand, then a weight function in $(0, \infty)$ can be defined at the points $q^{\pm k} \alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$.

PVI. If $x = q^{-1}a$ is a root of $\sigma_2(x, q)$, as we have already discussed, ρ is zero at $q^k a$ for $k \in \mathbb{N}_0$. Therefore, for $a > 0$ a q -weight function can exist in $(q^{-1}a, \infty)$ supported on the set of points $\{q^{-k}a\}_{k \in \mathbb{N}}$. An OPS does not exist if $a < 0$. Finally, if $a = 0$ it is possible to find a ρ on $(0, \infty)$ supported at the points $q^{\pm k} \alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$.

PVII. Finally, we consider the possibility of satisfying the BC

$$\lim_{a \rightarrow -\infty} \sigma_1(a, q) \rho(a, q) a^k = 0$$

in the limiting case as $a \rightarrow -\infty$. If this condition holds a weight function and, hence, an OPS w.r.t. a measure supported on the set of points $\{\pm q^{\pm k} \alpha\}_{k \in \mathbb{N}_0}$, for arbitrary $\alpha > 0$, can be defined.

The aforementioned considerations are expressible as a theorem.

Theorem 2.3 *Let $a_1(q)$, $b_1(q)$ and $a_2(q)$, $b_2(q)$ denote the zeros of $\sigma_1(x, q)$ and $\sigma_2(x, q)$, respectively. Let ρ be a bounded and non-negative function satisfying the q -Pearson equation (1.5) as well as the BCs (2.5) or (2.7). Then ρ is a desired weight function for the polynomial solutions $P_n(x, q)$ of (1.1) only in the following cases:*

1. *Let $a < 0 < b$, where $a = a_1(q)$ and $b = b_1(q)$. Then ρ is supported on $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^k b\}_{k \in \mathbb{N}_0}$ and*

$$\int_{a_1(q)}^{b_1(q)} P_n(x, q) P_m(x, q) \rho(x, q) d_q x = d_n^2(q) \delta_{mn}. \quad (2.11)$$

where the q -Jackson integral is of type (2.2).

2. *Let $a = 0 < b$, where $b = a_1(q)$. Then ρ is supported on the set of points $\{q^k b\}_{k \in \mathbb{N}_0}$ in $(0, b]$ and*

$$\int_0^{a_1(q)} P_n(x, q) P_m(x, q) \rho(x, q) d_q x = d_n^2(q) \delta_{mn}, \quad (2.12)$$

where the q -Jackson integral is of type (2.1).

3. *Let $0 < a < b$, where $a = a_2(q)$ and $b = q^{-1}a_1(q)$. Then ρ is supported on the finite set of points $\{q^{-k}a\}_{k=0}^N$ when $q^{-N-1}a = b$ and*

$$\int_{a_2(q)}^{q^{-1}a_1(q)=q^{-N-1}a_2(q)} P_n(x, q) P_m(x, q) \rho(x, q) d_{q^{-1}} x = d_n^2(q) \delta_{mn}, \quad (2.13)$$

which is the finite sum of the form

$$\int_{a_2(q)}^{q^{-N-1}a_2(q)} [\cdot] d_{q^{-1}} x = (1 - q^{-1}) a_2(q) \sum_{k=0}^N P_n(q^{-k} a_2(q), q) P_m(q^{-k} a_2(q), q) \rho(q^{-k} a_2(q), q).$$

4. Let $a = a_1(q) < 0$ and $b \rightarrow \infty$. Then ρ is supported on the set $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^{\mp k} a\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ and

$$\int_{a_1(q)}^{\infty} P_n(x, q) P_m(x, q) \rho(x, q) d_q x := \int_{a_1(q)}^0 [\cdot] d_q x + \int_0^{\infty} [\cdot] d_q x = d_n^2(q) \delta_{mn}, \quad (2.14)$$

where the first q -Jackson integral is of type (2.1) and the second one is of type (2.3), respectively.

5. Let $a = a_2(q) > 0$ and $b \rightarrow \infty$. Then ρ is supported on the set of points $\{q^{-k} a\}_{k \in \mathbb{N}_0}$ in $[a, \infty)$ and

$$\int_{a_2(q)}^{\infty} P_n(x, q) P_m(x, q) \rho(x, q) d_{q^{-1}} x = d_n^2(q) \delta_{mn}, \quad (2.15)$$

where the q^{-1} -Jackson integral is of type (2.4).

6. Let $a = 0$ and $b \rightarrow \infty$. Then ρ is supported on the set of points $\{q^{\pm k} \alpha\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ and

$$\int_0^{\infty} P_n(x, q) P_m(x, q) \rho(x, q) d_q x = d_n^2(q) \delta_{mn}, \quad (2.16)$$

where the q -Jackson integral is of type (2.3).

7. Let $a \rightarrow -\infty$ and $b \rightarrow \infty$. Then ρ is supported on the set of points $\{\mp q^{\pm k} \alpha\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ and

$$\int_{-\infty}^{\infty} P_n(x, q) P_m(x, q) \rho(x, q) d_q x = d_n^2(q) \delta_{mn}, \quad (2.17)$$

where the bilateral q -Jackson integral is of type (2.3).

Before starting our analysis, let us mention that in accordance with [5, 25, 29] the q -polynomials can be classified by means of the degrees of the polynomial coefficients σ_1 and σ_2 and the fact that either $\sigma_1(0, q)\sigma_2(0, q) \neq 0$ or $\sigma_1(0, q) = \sigma_2(0, q) = 0$. Therefore, we can define two classes, namely, the non-zero (\emptyset) and zero (0) classes corresponding to the cases where $\sigma_1(0, q)\sigma_2(0, q) \neq 0$ and $\sigma_1(0, q) = \sigma_2(0, q) = 0$, respectively (this is a consequence of the fact that $\sigma_2(0, q) = q\sigma_1(0, q)$, i.e., σ_1 and σ_2 both have the same constant terms). In each class we consider all possible degrees of the polynomial coefficients $\sigma_1(x, q)$ and $\sigma_2(x, q)$ as shown in [25, page 182]. We follow the notation introduced in [5, 25], i.e., the statement \emptyset -Laguerre/Jacobi implies that $\deg \sigma_2 = 1$, $\deg \sigma_1 = 2$, and $\sigma_1(0, q)\sigma_2(0, q) \neq 0$ and the statement 0-Jacobi/Laguerre indicates that $\deg \sigma_2 = 2$, $\deg \sigma_1 = 1$ and $\sigma_1(0, q) = \sigma_2(0, q) = 0$.

In the following we use the Taylor polynomial expansion for the coefficients

$$\begin{aligned} \tau(x, q) &= \tau'(0, q)x + \tau(0, q), \quad \tau'(0, q) \neq 0, \\ \sigma_1(x, q) &= \frac{1}{2}\sigma_1''(0, q)x^2 + \sigma_1'(0, q)x + \sigma_1(0, q) = \frac{1}{2}\sigma_1''(0, q)[x - a_1(q)][x - b_1(q)], \\ \sigma_2(x, q) &= \frac{1}{2}\sigma_2''(0, q)x^2 + \sigma_2'(0, q)x + \sigma_2(0, q) = \frac{1}{2}\sigma_2''(0, q)[x - a_2(q)][x - b_2(q)]. \end{aligned} \quad (2.18)$$

Theorem 2.4 (Classification of the OPS of the q -Hahn class [5, 25]) *All orthogonal polynomial solutions of the q -difference equations (1.1) and (1.3) can be classified as follows:*

1. \emptyset -**Jacobi/Jacobi** polynomials where $\sigma_2''(0, q) \neq 0$ and $\sigma_1''(0, q) \neq 0$ with $\sigma_1(0, q)\sigma_2(0, q) \neq 0$.
2. \emptyset -**Jacobi/Laguerre** polynomials where $\sigma_2''(0, q) \neq 0$ and $\sigma_1''(0, q) = 0$, $\sigma_1'(0, q) \neq 0$ with $\sigma_1(0, q)\sigma_2(0, q) \neq 0$.
3. \emptyset -**Jacobi/Hermite** polynomials where $\sigma_2''(0, q) \neq 0$, $\sigma_1''(0, q) = 0$, $\sigma_1'(0, q) = 0$ and $\sigma_1(0, q) \neq 0$ with $\sigma_1(0, q)\sigma_2(0, q) \neq 0$.

4. \emptyset -**Laguerre/Jacobi** polynomials where $\sigma_2''(0, q) = 0$, $\sigma_2'(0, q) \neq 0$ and $\sigma_1''(0, q) \neq 0$ with $\sigma_1(0, q)\sigma_2(0, q) \neq 0$.
5. \emptyset -**Hermite/Jacobi** polynomials where $\sigma_2''(0, q) = 0$, $\sigma_2'(0, q) = 0$, $\sigma_2(0, q) \neq 0$ and $\sigma_1''(0, q) \neq 0$ with $\sigma_1(0, q)\sigma_2(0, q) \neq 0$.
6. **0-Jacobi/Jacobi** polynomials where $\sigma_2''(0, q) \neq 0$, $\sigma_2'(0, q) \neq 0$ and $\sigma_1''(0, q) \neq 0$, $\sigma_1'(0, q) \neq 0$ with $\sigma_2(0, q) = \sigma_1(0, q) = 0$.
7. **0-Jacobi/Laguerre** polynomials where $\sigma_2''(0, q) \neq 0$, $\sigma_2'(0, q) \neq 0$ and $\sigma_1''(0, q) = 0$, $\sigma_1'(0, q) \neq 0$ with $\sigma_2(0, q) = \sigma_1(0, q) = 0$.
8. **0-Bessel/Jacobi** polynomials where $\sigma_2''(0, q) \neq 0$, $\sigma_2'(0, q) = 0$ and $\sigma_1''(0, q) \neq 0$, $\sigma_1'(0, q) \neq 0$ with $\sigma_2(0, q) = \sigma_1(0, q) = 0$.
9. **0-Bessel/Laguerre** polynomials where $\sigma_2''(0, q) \neq 0$, $\sigma_2'(0, q) = 0$ and $\sigma_1''(0, q) = 0$, $\sigma_1'(0, q) \neq 0$ with $\sigma_2(0, q) = \sigma_1(0, q) = 0$.
10. **0-Laguerre/Jacobi** polynomials where $\sigma_2''(0, q) = 0$, $\sigma_2'(0, q) \neq 0$ and $\sigma_1''(0, q) \neq 0$, $\sigma_1'(0, q) \neq 0$ with $\sigma_2(0, q) = \sigma_1(0, q) = 0$.

3 The q -weight function

The explicit form of a q -weight function can be deduced by means of Theorem 3.1.

Theorem 3.1 *Let f satisfy the difference equation*

$$\frac{f(qx; q)}{f(x; q)} = \frac{a(x; q)}{b(x; q)}, \quad (3.1)$$

such that the limits $\lim_{x \rightarrow 0} f(x; q) = f(0, q)$ and $\lim_{x \rightarrow \infty} f(x; q) = f(\infty, q)$ exist, where $a(x; q)$ and $b(x; q)$ are definite functions. Then $f(x; q)$ admits the two q -integral representations

$$f(x, q) = f(0, q) \exp \left[\int_0^x \frac{1}{(q-1)t} \ln \left[\frac{a(t, q)}{b(t, q)} \right] d_q t \right] \quad (3.2)$$

and

$$f(x, q) = f(\infty, q) \exp \left[\int_x^\infty \frac{1}{(1-q^{-1})t} \ln \left[\frac{a(t, q)}{b(t, q)} \right] d_{q^{-1}} t \right] \quad (3.3)$$

provided that the integrals are convergent.

Proof Taking the logarithms of both sides of (3.1), multiplying by $1/(q-1)t$ and then integrating from 0 to x , we have

$$\int_0^x \frac{1}{(q-1)t} \ln \left[\frac{f(qt, q)}{f(t, q)} \right] d_q t = \int_0^x \frac{1}{(q-1)t} \ln \left[\frac{a(t, q)}{b(t, q)} \right] d_q t.$$

The l.h.s. is expressible as

$$\begin{aligned} \int_0^x \frac{1}{(q-1)t} \ln \left[\frac{f(qt, q)}{f(t, q)} \right] d_q t &= \lim_{n \rightarrow \infty} \sum_{j=0}^n [\ln(f(q^j x, q)) - \ln(f(q^{j+1} x, q))] \\ &= \ln[f(x, q)] - \ln[f(0, q)], \end{aligned}$$

which completes the proof, on using the fact that $f(q^{n+1}x, q) \rightarrow f(0, q)$ as $n \rightarrow \infty$ for $0 < q < 1$. The second representation (3.3) can be proven in a similar way. ■

This theorem can be used to derive the q -weight functions for every σ_1 and σ_2 . However, here we take into account the quadratic coefficients leading to \emptyset -Jacobi/Jacobi and 0-Jacobi/Jacobi cases. The results, some of which can be found in [5], are stated by the next theorem.

Theorem 3.2 *In the \emptyset -Jacobi/Jacobi case, let $\sigma_1(x, q)$ and $\sigma_2(x, q)$ be of forms (2.18) in which $\sigma_1''(0, q)a_1(q)b_1(q) \neq 0$ and $\sigma_2''(0, q)a_2(q)b_2(q) \neq 0$. And let, in 0-Jacobi/Jacobi case, $b_1(q) = b_2(q) = 0$, $\sigma_1''(0, q) \neq 0$ and $\sigma_2''(0, q) \neq 0$. Then a solution $\rho(x, q)$ of q -Pearson equation (1.5) is expressible in the equivalent forms shown in Table 1.*

Table 1: Expressions for the q -weight function $\rho(x, q)$ in the Jacobi/Jacobi cases

| \emptyset -Jacobi/Jacobi case | |
|---------------------------------|---|
| 1. | $\frac{(a_1^{-1}qx, b_1^{-1}qx; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$ |
| 2. | $ x ^\alpha \frac{(b_1^{-1}qx, a_2q/x; q)_\infty}{(a_1/x, b_2^{-1}x; q)_\infty}$ where $q^\alpha = \frac{q^{-2}\sigma_2''(0, q)b_2}{\sigma_1''(0, q)b_1}$ |
| 0-Jacobi/Jacobi case | |
| 3. | $ x ^\alpha \frac{(a_1^{-1}qx; q)_\infty}{(a_2^{-1}x; q)_\infty}$ where $q^\alpha = \frac{q^{-2}\sigma_2''(0, q)a_2}{\sigma_1''(0, q)a_1}$ |
| 4. | $ x ^\alpha \sqrt{x^{\log_q x - 1}} (qa_1^{-1}x, qa_2/x; q)_\infty$ where $q^\alpha = \frac{q^{-2}\sigma_2''(0, q)}{\sigma_1''(0, q)a_1}$ |

Proof We start proving the first expression in Table 1. Keeping in mind that $q^{-1}\sigma_2(0, q) = \sigma_1(0, q)$ and that $q^{-1}\sigma_2''(0, q)a_2(q)b_2(q) = \sigma_1''(0, q)a_1(q)b_1(q)$ we have from (1.5)

$$\frac{\rho(qx, q)}{\rho(x, q)} = \frac{q^{-1}\sigma_2''(0, q)[x - a_2(q)][x - b_2(q)]}{\sigma_1''(0, q)[qx - a_1(q)][qx - b_1(q)]} = \frac{[1 - a_2^{-1}(q)x][1 - b_2^{-1}(q)x]}{[1 - a_1^{-1}(q)qx][1 - b_1^{-1}(q)qx]} \quad (3.4)$$

which gives

$$\rho(x, q) = \rho(0, q) \exp \left\{ \int_0^x \frac{1}{(q-1)t} \left[\ln(1 - a_2^{-1}t) + \ln(1 - b_2^{-1}t) - \ln(1 - a_1^{-1}qt) - \ln(1 - b_1^{-1}qt) \right] d_q t \right\}$$

on using (3.2). By definition (2.1) of the q -integral, we first obtain

$$\rho(x, q) = \rho(0, q) \exp \left\{ \ln \prod_{k=0}^{\infty} (1 - a_1^{-1}q^{k+1}x)(1 - b_1^{-1}q^{k+1}x) - \ln \prod_{k=0}^{\infty} (1 - a_2^{-1}q^k x)(1 - b_2^{-1}q^k x) \right\}$$

and, therefore,

$$\rho(x, q) = \rho(0, q) \frac{(a_1^{-1}qx, b_1^{-1}qx; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}, \quad \rho(0, q) \neq 0. \quad (3.5)$$

This implies that $a_1(q)q^{-1-k}$ and $b_1(q)q^{-1-k}$ for $k \geq 0$ are zeros of ρ . Furthermore, $a_2(q)q^{-j}$ and $b_2(q)q^{-j}$ for $j \geq 0$ stand for the simple poles of ρ . Note here that $\rho(0, q)$ can be made unity, and $a_1(q)$, $b_1(q)$, $a_2(q)$ and $b_2(q)$ are non-zero constants. Therefore the solution in (3.5) is continuous everywhere except at the simple poles.

To show 4., we rewrite the q -Pearson equation in the form

$$\frac{\rho(qx, q)}{\rho(x, q)} = \frac{ax[1 - a_2(q)/x]}{[1 - a_1^{-1}(q)qx]}, \quad a = \frac{q^{-2}\sigma_2''(0, q)}{\sigma_1''(0, q)a_1(q)} \quad (3.6)$$

and assume that ρ is a product of three functions $\rho(x, q) = f(x, q)g(x, q)h(x, q)$. Hence, if f , g and h are solutions of

$$\frac{f(qx, q)}{f(x, q)} = ax, \quad \frac{g(qx, q)}{g(x, q)} = \frac{1}{[1 - a_1^{-1}(q)qx]} \quad \text{and} \quad \frac{h(qx, q)}{h(x, q)} = \left[1 - \frac{a_2(q)}{x}\right] \quad (3.7)$$

respectively, then $\rho = fgh$ is a solution of (3.6). A solution of (3.7) for $f(x, q)$ is of the form $f(x, q) = |x|^\alpha H^{(1)}(x)$, which can be verified by direct substitution. Here, the function $H^{(\beta)}(x) = \sqrt{x^{\log_q x} - \beta}$ with $\beta \neq 0$ was first defined in [5], and $\alpha \neq 0$ is such that $q^\alpha = a$. The equation in (3.7) for $g(x, q)$ can be solved in a way similar to that of (3.4). So we find that $g(x, q) = g(0, q)(a_1^{-1}qx; q)_\infty$, where $g(0, q) = 1$. The expression (3.2) is not suitable in finding $h(x, q)$ which gives a divergent infinite product. Instead, we employ (3.3) so that the equation for $h(x, q)$ becomes

$$\frac{h(q^{-1}x, q)}{h(x, q)} = \frac{1}{[1 - qa_2(q)/x]}$$

whose solution is of the form $h(x, q) = h(\infty, q)(qa_2/x; q)_\infty$, where $h(\infty, q)$ can be taken again as unity without loss of generality. Clearly $h(x, q)$ is uniformly convergent in any compact subset of the complex plane that does not contain the point at the origin. Moreover, the product converges to an arbitrary constant c , which has been set to unity, as $x \rightarrow \infty$. Thus,

$$\rho(x, q) = f(x, q)g(x, q)h(x, q) = |x|^\alpha \sqrt{x^{\log_q x} - 1} (qa_1^{-1}x, qa_2/x; q)_\infty.$$

In order to obtain the expressions 2 and 3 in Table 1 for the weight function we use the same procedure as before, but starting from the q -Pearson equation written in the forms

$$\frac{\rho(qx, q)}{\rho(x, q)} = a \frac{[1 - a_2(q)/x][1 - b_2^{-1}(q)x]}{[1 - a_1(q)q^{-1}/x][1 - b_1^{-1}(q)qx]}, \quad a = \frac{q^{-2}\sigma_2''(0, q)b_2(q)}{\sigma_1''(0, q)b_1(q)} \quad (3.8)$$

and

$$\frac{\rho(qx, q)}{\rho(x, q)} = a \frac{[1 - a_2^{-1}(q)x]}{[1 - a_1^{-1}(q)qx]}, \quad a = \frac{q^{-2}\sigma_2''(0, q)a_2(q)}{\sigma_1''(0, q)a_1(q)}, \quad (3.9)$$

respectively. This completes the proof. ■

Remark 3.3 Notice that for getting the expressions of the weight function we have used the q -Pearson equation rewritten in different forms, namely (3.4), (3.6), (3.8) and (3.9), and different solution procedure in each case, therefore, it is not surprising that ρ has several equivalent representations displayed in Table 1. However, they all satisfy the same linear equation and, therefore, they differ only by a multiplicative constant.

For the sake of the completeness, let us obtain the analytic representations of q -weight functions satisfying (1.5) for the other cases.

Theorem 3.4 Let σ_1 and σ_2 be polynomials of at most 2nd degree in x as the form (2.18). Then a solution $\rho(x, q)$ of q -Pearson equation (1.5) for each \emptyset -Jacobi/Laguerre, \emptyset -Jacobi/Hermite, \emptyset -Laguerre/Jacobi, \emptyset -Hermite/Jacobi, 0-Jacobi/Laguerre, 0-Bessel/Jacobi, 0-Bessel/Laguerre and 0-Laguerre/Jacobi case is expressible in the equivalent forms shown in Table 2.

Proof The proof is similar to the previous one. That is, to obtain the second formula for the \emptyset -Jacobi/Laguerre family we rewrite the q -Pearson equation (1.5) in the form

$$\frac{\rho(qx, q)}{\rho(x, q)} = ax \frac{[1 - a_2/x][1 - b_2/x]}{[1 - a_1q^{-1}/x]}, \quad a = \frac{q^{-2\frac{1}{2}}\sigma_2''(0, q)}{\sigma_1'(0, q)}$$

and then apply the same procedure described in the proof of the previous theorem. ■

Table 2: Expressions for the q -weight function $\rho(x, q)$ for the other cases

| | |
|------------------------------|--|
| \emptyset -Jacobi/Laguerre | 1. $\frac{(a_1^{-1}qx; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$ 2. $ x ^\alpha \sqrt{x^{\log_q x - 1}} \frac{(a_2q/x, b_2q/x; q)_\infty}{(a_1/x; q)_\infty}$, $q^\alpha = \frac{\frac{1}{2}\sigma_2''(0, q)q^{-2}}{\sigma_1'(0, q)}$ 3. $ x ^\alpha x^{\log_q x} (qa_1^{-1}x, qa_2/x, qb_2/x; q)_\infty$, $q^\alpha = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\sigma_1'(0, q)a_1}$ |
| \emptyset -Jacobi/Hermite | 1. $\frac{1}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$ 2. $ x ^\alpha x^{\log_q x - 1} (a_2q/x, b_2q/x; q)_\infty$, $q^\alpha = \frac{\frac{1}{2}\sigma_2''(0, q)q^{-1}}{\sigma_1(0, q)}$ |
| \emptyset -Laguerre/Jacobi | 1. $\frac{(a_1^{-1}qx, b_1^{-1}qx; q)_\infty}{(a_2^{-1}x; q)_\infty}$ 2. $ x ^\alpha \frac{(qa_2/x, qb_1^{-1}x; q)_\infty}{(a_1/x; q)_\infty}$, $q^\alpha = -\frac{q^{-2}\sigma_2'(0, q)}{\frac{1}{2}\sigma_1''(0, q)b_1}$ |
| \emptyset -Hermite/Jacobi | $(a_1^{-1}qx, b_1^{-1}qx; q)_\infty$ |
| 0-Jacobi/Laguerre | 1. $ x ^\alpha \frac{1}{(a_2^{-1}x; q)_\infty}$, $q^\alpha = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)a_2}{\sigma_1'(0, q)}$ 2. $ x ^\alpha \sqrt{x^{\log_q x - 1}} (qa_2/x; q)_\infty$, $\frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\sigma_1'(0, q)}$ |
| 0-Bessel/Jacobi | $ x ^\alpha \sqrt{x^{\log_q x - 1}} (a_1^{-1}qx; q)_\infty$, $q^\alpha = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\frac{1}{2}\sigma_1''(0, q)a_1}$ |
| 0-Bessel/Laguerre | $ x ^\alpha \sqrt{x^{\log_q x - 1}}$, $q^\alpha = \frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\sigma_1'(0, q)}$ |
| 0-Laguerre/Jacobi | $ x ^\alpha (a_1^{-1}qx; q)_\infty$, $q^\alpha = -\frac{q^{-2}\sigma_2'(0, q)}{\frac{1}{2}\sigma_1''(0, q)a_1}$ |

4 The orthogonality of q -polynomials: the Jacobi/Jacobi cases

The rational function on the r.h.s. of the q -Pearson equation (1.5) has been examined in detail. Since it is the ratio of two polynomials σ_1 and σ_2 of at most second degree, we deal with a definite rational function having at most two zeros and two poles. In the analysis of the unknown quantity $\rho(qx, q)/\rho(x, q)$ on the l.h.s. of (1.5), we sketch roughly its graph by using every possible form of the definite rational function in question. In particular, we split the x -interval into subintervals according to whether $\rho(qx, q)/\rho(x, q) < 1$ or $\rho(qx, q)/\rho(x, q) > 1$, which yields valuable information about the monotonicity of $\rho(x, q)$. Other significant properties of ρ are provided by the asymptotes, if there exist any, of $\rho(qx, q)/\rho(x, q)$. A full analysis along these lines is sufficient for a complete characterization of the orthogonal q -polynomials. A similar characterization is made in a very recent book [19] based on the three-term recursion and the Favard theorem.

Here, in this section, we discuss only the cases in which both σ_1 and σ_2 are of second degree, i.e., the \emptyset - and 0-Jacobi/Jacobi cases.

4.1 The non-zero case

Let the coefficients σ_1 and σ_2 be quadratic polynomials in x such that $\sigma_1(0, q)\sigma_2(0, q) \neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)[x - a_1(q)][x - b_1(q)]$ then, from (1.4),

$$\sigma_2(x, q) = \left[\frac{1}{2}\sigma_1''(0, q) + (1 - q^{-1})\tau'(0, q) \right] qx^2 - \left[\frac{1}{2}\sigma_1''(0, q)(a_1 + b_1) - (1 - q^{-1})\tau(0, q) \right] qx + \frac{1}{2}q\sigma_1''(0, q)a_1b_1$$

where $\frac{1}{2}\sigma_1''(0, q) + (1 - q^{-1})\tau'(0, q) \neq 0$ by hypothesis. Then q -Pearson equation (1.5) takes the form

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)} = \left[1 + \frac{(1 - q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] \frac{[x - a_2(q)][x - b_2(q)]}{[qx - a_1(q)][qx - b_1(q)]} \quad (4.1)$$

provided that the discriminant denoted by Δ_q ,

$$\Delta_q := \left[a_1(q) + b_1(q) - \frac{(1 - q^{-1})\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right]^2 - 4a_1(q)b_1(q) \left[1 + \frac{(1 - q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right],$$

of the quadratic polynomial in the nominator of $f(x, q)$ in (4.1) is non-zero. Here $x = a_2$ and $x = b_2$ denote the zeros of f , and they are constant multiples of the roots of $\sigma_2(x, q)$.

We see that the lines $x = q^{-1}a_1$ and $x = q^{-1}b_1$ stand for the vertical asymptotes of $f(x, q)$ and the point $y = 1$ is always its y -intercept since $\sigma_2(0, q) = q\sigma_1(0, q)$. Moreover, the locations of the zeros of f are determined by the straightforward lemma.

Lemma 4.1 *Define the parameter*

$$\Lambda_q = \frac{1}{q^2} \left[1 + \frac{(1 - q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] \neq 0 \quad (4.2)$$

so that the line $y = \Lambda_q$ denotes the horizontal asymptote of $f(x, q)$. Then we encounter the following cases for the roots of the equation $f(x, q) = 0$.

Case 1. *If $\Lambda_q > 0$ and $a_1(q) < 0 < b_1(q)$, f has two real and distinct roots with opposite signs.*

Case 2. *If $\Lambda_q > 0$ and $0 < a_1(q) < b_1(q)$, there exist three possibilities*

- (a) *if $\Delta_q > 0$, f has two real roots with the same signs*
- (b) *if $\Delta_q = 0$, f has a double root*
- (c) *if $\Delta_q < 0$, f has a pair of complex conjugate roots.*

Case 3. *If $\Lambda_q < 0$ and $a_1(q) < 0 < b_1(q)$, there exist three possibilities*

- (a) *if $\Delta_q > 0$, f has two real roots with the same signs*
- (b) *if $\Delta_q = 0$, f has a double root*
- (c) *if $\Delta_q < 0$, f has a pair of complex conjugate roots.*

Case 4. *If $\Lambda_q < 0$ and $0 < a_1(q) < b_1(q)$, f has two real distinct roots with opposite signs.*

From (4.1) it is clear that we need to consider the cases $\Lambda_q > 1$ and $0 < \Lambda_q < 1$ separately. Now, our strategy consists of sketching first the graphs of $f(x, q)$ depending on all possible relative positions of the zeros of σ_1 and σ_2 . To obtain the behaviours of q -weight functions ρ from the graphs of $f(x, q) = \rho(qx, q)/\rho(x, q)$, we divide the real line into subintervals in which ρ is either monotonic decreasing or increasing. We take into consideration only the subintervals where $\rho > 0$. Note that if ρ is initially positive then we have $\rho > 0$ everywhere in an interval where $\rho(qx, q)/\rho(x, q) > 0$. Then we find suitable intervals in cooperation with Theorem 2.3.

In Figure 1A, the intervals $(q^{-1}a_1, a_2)$ and $(b_2, q^{-1}b_1)$ are rejected immediately since f is negative there. The subinterval (a_2, b_2) should also be rejected in which $\rho = 0$ by **PII**. For the same reason $(q^{-1}b_1, \infty)$ and $(-\infty, q^{-1}a_1)$, by symmetry, are not suitable by **PV**. Therefore, an OPS fails to exist.

Let us analyse the problem presented in Figure 1B. The positivity of ρ implies that the intervals $(q^{-1}a_1, a_2)$ and $(q^{-1}b_1, b_2)$ should be eliminated. With the transformation $x = -t$, we eliminate also $(-\infty, q^{-1}a_1)$ by **PV**. The interval $(a_2, q^{-1}b_1)$ is not suitable too, by **PIII**. So it remains only (b_2, ∞) to examine which coincides with the 5th case in Theorem 2.3. Since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > b_2(q)$, then ρ is increasing on (b_2, x_0) and decreasing on (x_0, ∞) . As is shown from the figure f has a finite limit as $x \rightarrow +\infty$ so that we could have the case $\rho \rightarrow 0$ as $x \rightarrow \infty$. Even

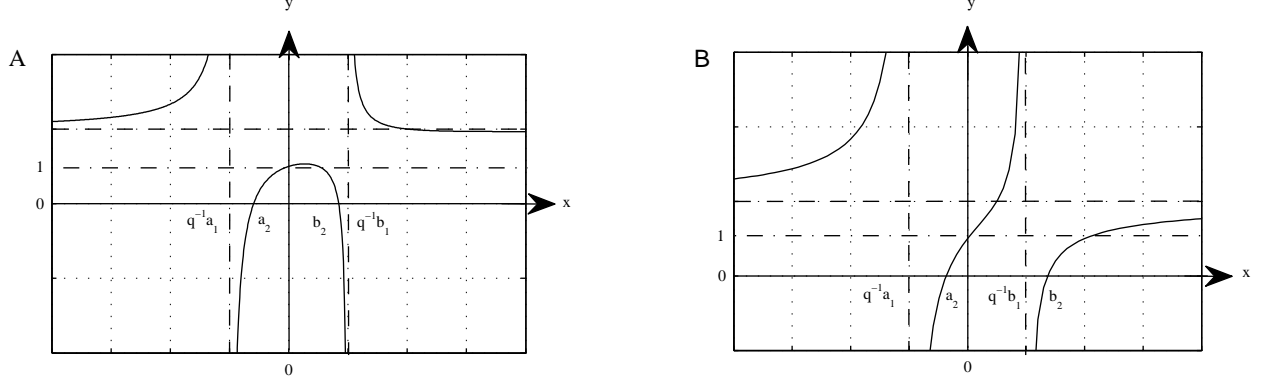


Figure 1: The graph of $f(x, q)$ in **Case 1** with $\Lambda_q > 1$. In A, the zeros are in order $q^{-1}a_1 < a_2 < 0 < b_2 < q^{-1}b_1$, and in B, $q^{-1}a_1 < a_2 < 0 < q^{-1}b_1 < b_2$.

if $\rho \rightarrow 0$ as $x \rightarrow \infty$, we must show also that $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ to satisfy the BC. In fact, instead of the usual q -Pearson equation we have to consider the equation

$$g(x, q) := \frac{\sigma_1(qx, q)\rho(qx, q)(qx)^k}{\sigma_1(x, q)\rho(x, q)x^k} = q^k \frac{\sigma_1(x, q) + (1 - q^{-1})x\tau(x, q)}{\sigma_1(x, q)} = q^k \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(x, q)} \quad (4.3)$$

in case of an infinite interval, what we call it here the *extended* q -Pearson equation to determine the behaviour of the quantity $\sigma_1(x, q)\rho(x, q)x^k$ as $x \rightarrow \infty$, which has been easily derived from (1.5). It is obvious that the extended q -Pearson equation is the difference equation not for the weight function $\rho(x, q)$ but for $\sigma_1(x, q)\rho(x, q)x^k$.

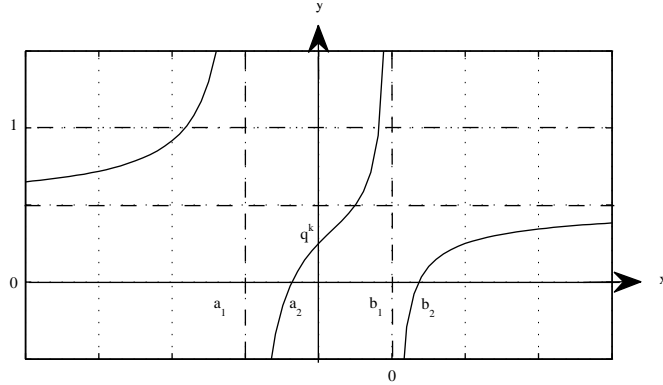


Figure 2: The graph of $g(x, q)$ corresponding to Figure 1B.

In Figure 2 we draw the graph of a typical g for some $0 < q < 1$, where k is large enough. From this figure we see that $g < 1$ for $x > b_2$ so that $\sigma_1(x, q)\rho(x, q)x^k$ does not vanish at ∞ since it is increasing as x increases. Thus we cannot find a weight function ρ on (b_2, ∞) .

From Figure 3A, we first eliminate the intervals $(a_2, q^{-1}a_1)$ and $(q^{-1}b_1, b_2)$ because of the positivity of ρ . The interval (b_2, ∞) coincides again with the 5th case in Theorem 2.3. However, $f(x, q) < 1$ on this interval so that ρ is increasing on (b_2, ∞) which implies that ρ can not vanish as $x \rightarrow \infty$. Thus $\sigma_1(x, q)\rho(x, q)x^k$ is never zero as $x \rightarrow \infty$ for some $k \in \mathbb{N}_0$. The same is true for $(-\infty, a_2)$ by symmetry. For the last subinterval $(q^{-1}a_1, q^{-1}b_1)$, we face the 1th case in Theorem 2.3. Since $\rho(qx, q)/\rho(x, q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qx, q)/\rho(x, q) \rightarrow \infty$, and hence $\rho \rightarrow 0$, as $x \rightarrow q^{-1}a_1^+$ and $x \rightarrow q^{-1}b_1^-$. As a result, the typical shape of ρ is shown in Figure 4 assuming a positive initial value of ρ in each subinterval. Then, an OPS with such a weight function in Figure 4 supported on the union of set of

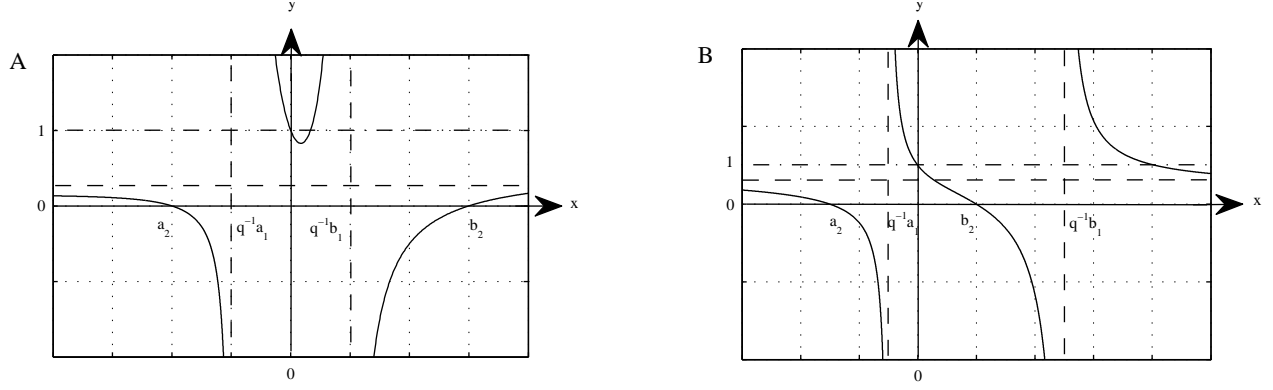


Figure 3: The graph of $f(x, q)$ in **Case 1** with $0 < \Lambda_q < 1$. In A, the zeros are in order $a_2 < q^{-1}a_1 < 0 < q^{-1}b_1 < b_2$ and in B, $a_2 < q^{-1}a_1 < 0 < b_2 < q^{-1}b_1$.

points $\{q^k a_1(q)\}_{k \in \mathbb{N}_0}$ and $\{q^k b_1(q)\}_{k \in \mathbb{N}_0}$ exists (see Theorem 2.3-1). This OPS can be stated in the Theorem 4.2.

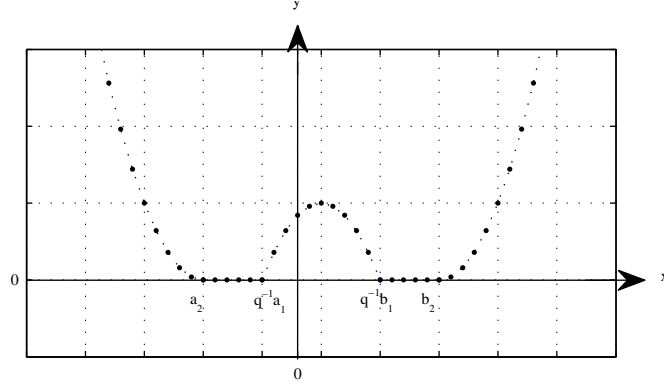


Figure 4: The graph of $\rho(x, q)$ associated with the case in Figure 3A.

Theorem 4.2 Consider the case where $a_2 < a_1 < 0 < b_1 < b_2$ and $0 < q^2 \Lambda_q < 1$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see Eq. 1 in Table 1)

$$\rho(x, q) = \frac{(qa^{-1}x, qb^{-1}x; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}, \quad (4.4)$$

supported on $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^k b\}_{k \in \mathbb{N}_0}$, (see (2.11) of Theorem 2.3-1).

The OPS in Theorem 4.2 coincides with the case VIIa1 in Chapter 10 of [19, pages 292 and 318]. In fact, a typical example of this family is the big q -Jacobi polynomials $P_n(x; a, b, c; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-2}(x - a_1)(x - b_1), \quad \sigma_2(x, q) = abq(x - a_2)(x - b_2),$$

$$\tau(x, q) = \frac{1 - abq^2}{(1 - q)q}x + \frac{a(bq - 1) + c(aq - 1)}{1 - q} \quad \text{and} \quad \lambda_n(q) = q^{-n}[n]_q \frac{1 - abq^{n+1}}{q - 1} \quad (4.5)$$

where $a_1 = cq$, $b_1 = aq$, $a_2 = b^{-1}c$ and $b_2 = 1$. The conditions $0 < q^2 \Lambda_q < 1$ and $a_2 < a_1 < 0 < b_1 < b_2$ give the known constraints $c < 0$, $0 < b < q^{-1}$ and $0 < a < q^{-1}$ on the parameters of $P_n(x; a, b, c; q)$

with orthogonality on $\{cq, cq^2, cq^3, \dots\} \cup \{\dots, aq^3, aq^2, aq\}$ in the sense (2.11) with

$$d_n^2 = (a-c)q(1-q) \frac{(q, abq^2, a^{-1}cq, ac^{-1}q; q)_\infty}{(aq, bq, cq, abc^{-1}q; q)_\infty} \frac{(q, abq; q)_n}{(abq, abq^2; q)_{2n}} (aq, bq, cq, abc^{-1}q; q)_n (-ac)^n q^{n(n+3)/2}.$$

It should be noted that the difference between these conditions and those of Figure 3 comes from the fact that we have considered not only the conditions on ρ but also on $\sigma_1\rho$ in Theorem 4.2. Finally, the analysis of the case in Figure 3B does not yield an OPS.

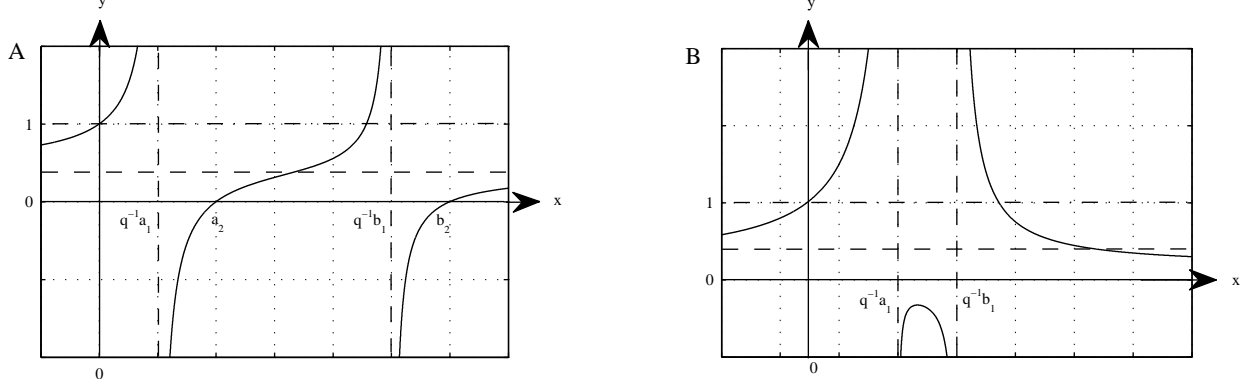


Figure 5: The graph of $f(x, q)$ in **Case 2** with $0 < \Lambda_q < 1$. In A, we have **Case 2(a)** with $0 < q^{-1}a_1 < a_2 < q^{-1}b_1 < b_2$. In B, we have **Case 2(c)** with $0 < q^{-1}a_1 < q^{-1}b_1$ and $a_2, b_2 \in \mathbb{C}$.

The case in Figure 5B is inappropriate to define an OPS. On the other hand, in Figure 5A, the intervals $(q^{-1}a_1, a_2)$ and $(q^{-1}b_1, b_2)$ are rejected by the positivity of ρ . The intervals $(-\infty, q^{-1}a_1)$ and (b_2, ∞) coincide with 4th, by symmetry, and 5th cases in Theorem 2.3. However, $f(x, q) < 1$ on $(-\infty, 0)$ and on (b_2, ∞) so that ρ is decreasing on $(-\infty, 0)$ and increasing on (b_2, ∞) which implies that ρ can not vanish as $x \rightarrow -\infty$ and $x \rightarrow \infty$. Thus $\sigma_1(x, q)\rho(x, q)x^k$ is never zero as $x \rightarrow -\infty$ and $x \rightarrow \infty$ for some $k \in \mathbb{N}_0$. The only possible interval for the case in Figure 5A is $(a_2, q^{-1}b_1)$ which corresponds to the 3th case of Theorem 2.3. Note that $\rho(qx, q)/\rho(x, q) = 1$ at $a_2 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$ and ρ is increasing on (a_2, x_0) and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qa_2, q) = 0$ and $\rho \rightarrow 0$ as $x \rightarrow q^{-1}b_1^-$ since $\rho(qa_2, q)/\rho(a_2, q) = 0$ and $\rho(qx, q)/\rho(x, q) \rightarrow \infty$ as $x \rightarrow q^{-1}b_1^-$, respectively. It is clear that the BCs are satisfied at $x = a_2$ and $x = q^{-1}b_1$. Thus we can find a suitable ρ on $[a_2, q^{-1}b_1)$ supported at the points $q^{-k}a_2$, $k = 0, 1, \dots, N$ where $q^{-N-1}a_2 = q^{-1}b_1$. Therefore, we state the following theorem.

Theorem 4.3 Consider the case where $0 < a_1 < a_2 < b_1 < b_2$ and $0 < q^2\Lambda_q < 1$. Let $a = a_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x, q)$ and $\sigma_1(qx, q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (see Eq. 2 in Table 1)

$$\rho(x, q) = |x|^t \frac{(\frac{qa}{x}, b^{-1}x; q)_\infty}{(\frac{a_1}{x}, b_2^{-1}x; q)_\infty}, \quad q^t = \frac{q^{-3}\sigma_2''(0, q)b_2}{\sigma_1''(0, q)b} \quad (4.6)$$

supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-3).

The OPS in Theorem 4.3 corresponds to the case IIIb9 in Chapter 11 of [19, page 366]. A well known example of this family is the q -Hahn polynomials satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-2}(x - a_1)(x - b_1), \quad \sigma_2(x, q) = \alpha\beta q(x - a_2)(x - b_2),$$

$$\tau(x, q) = \frac{1 - \alpha\beta q^2}{(1 - q)q}x + \frac{\alpha q^{-N} + \alpha\beta q - \alpha - q^{-N-1}}{1 - q} \quad \text{and} \quad \lambda_n(q) = -q^{-n}[n]_q \frac{1 - \alpha\beta q^{n+1}}{1 - q} \quad (4.7)$$

where $a_1 = \alpha q$, $b_1 = q^{-N}$, $a_2 = 1$ and $b_2 = \beta^{-1}q^{-N-1}$. The conditions $0 < q^2\Lambda_q < 1$ and $0 < a_1 < a_2 < b_1 < b_2$ give the conditions $0 < \alpha < q^{-1}$ and $0 < \beta < q^{-1}$ on the parameters of $Q_n(x; \alpha, \beta, N|q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$ in the sense (2.13) where

$$d_n^2 = \frac{(q, q^{N+1}, \beta^{-1}, \alpha^{-1}\beta^{-1}q^{-N-1}; q)_\infty}{(\alpha q, \beta q^{N+1}, \beta^{-1}q^{-N}, \alpha^{-1}\beta^{-1}q^{-1}; q)_\infty} \frac{(q, \alpha\beta q, \alpha q, q^{-N}, \beta q, \alpha\beta q^{N+2}; q)_n}{(\alpha\beta q, \alpha\beta q^2; q)_{2n}} (-\alpha q^{-N})^n q^{n(n+1)/2} (q^{-1} - 1). \quad (4.8)$$

In the literature, this relation is usually written as a finite sum [19, page 367].

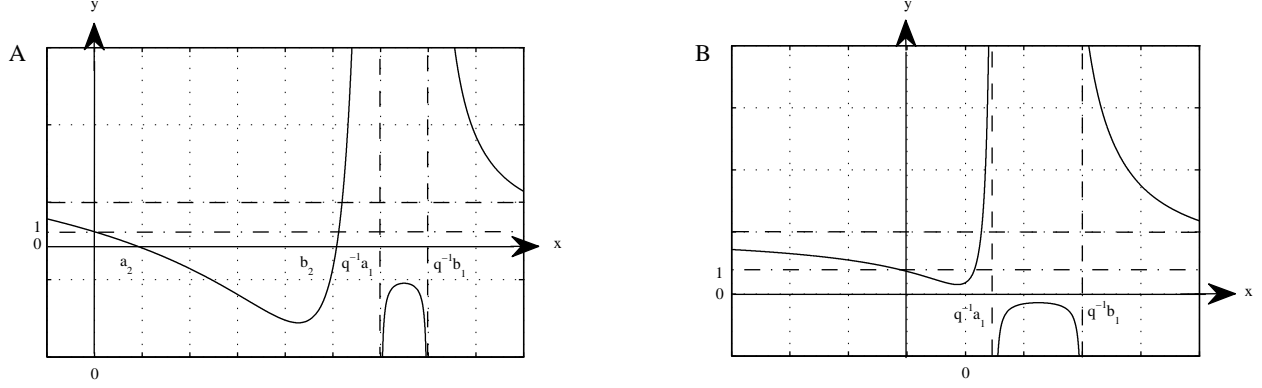


Figure 6: The graph of $f(x, q)$ in **Case 2** with $\Lambda_q > 1$. In A, we have **Case 2(a)** with $0 < a_2 < b_2 < q^{-1}a_1 < q^{-1}b_1$. In B, we have **Case 2(c)** with $0 < q^{-1}a_1 < q^{-1}b_1$ and $a_2, b_2 \in \mathbb{C}$.

In Figure 6A, the intervals (a_2, b_2) and $(q^{-1}a_1, q^{-1}b_1)$ are rejected by the positivity of ρ . We also eliminate the intervals $(-\infty, a_2)$ and $(q^{-1}b_1, \infty)$ due to **PVI**, by symmetry, and **PV**, respectively. The last interval $[b_2, q^{-1}a_1]$ coincides with the 3th case in Theorem 2.3. Notice that $f(x, q) = 1$ at $b_2 < x_0 = -\tau'(0, q)/\tau(0, q) < q^{-1}a_1$, then ρ is increasing on (b_2, x_0) with $\rho(qb_2, q) = 0$ since $\rho(qb_2, q)/\rho(b_2, q) = 0$ and decreasing on $(x_0, q^{-1}a_1)$ with $\rho \rightarrow 0$ as $x \rightarrow q^{-1}a_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. As a result, $[b_2, q^{-1}a_1]$ is an interval in which a desired ρ is defined on the supporting points $q^{-k}b_2(q)$ for $k = 1, 2, \dots, N$ such that $q^{-N-1}b_2 = q^{-1}a_1$. Notice that the BC (2.7) holds since $q^{-1}a_1$ and b_2 are one of the roots of $\sigma_1(qx, q)$ and $\sigma_2(x, q)$, respectively. As a significant remark, observe that the analysis is valid in the limiting cases $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$ as well. Hence, the resulting OPS is presented in Theorem 4.4. However, the case in Figure 6B does not give any OPS.

Theorem 4.4 Consider the case where $0 < a_2 \leq b_2 < a_1 \leq b_1$ and $q^2\Lambda_q > 1$. Let $a = b_2(q)$ and $b = q^{-1}a_1(q)$ be one of the zeros of $\sigma_2(x, q)$ and $\sigma_1(qx, q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (see Eq. 2 in Table 1)

$$\rho(x, q) = |x|^\ell \frac{(b^{-1}x, \frac{qa}{x}; q)_\infty}{(\frac{b_1}{x}, a_2^{-1}x; q)_\infty}, \quad q^\ell = \frac{q^{-3}\sigma_2''(0, q)a_2}{\sigma_1''(0, q)b}$$

supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-3).

An example of this family is again the q -Hahn polynomials $Q_n(x; \alpha, \beta, N|q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$. They satisfy (1.1) and (1.3) with the coefficients $\sigma_1(x, q) = q^{-2}(x - a_1)(x - b_1)$, $\sigma_2(x, q) = \alpha\beta q(x - a_2)(x - b_2)$, as in (4.7) but where now $a_1 = q^{-N}$, $b_1 = \alpha q$, $a_2 = \beta^{-1}q^{-N-1}$ and $b_2 = 1$. These polynomials satisfy the orthogonality relation (4.8) with the same d_n^2 but with a different choice of the parameters $\alpha \geq q^{-N-1}$ and $\beta \geq q^{-N-1}$ which comes from the conditions $0 < a_2 \leq b_2 < a_1 \leq b_1$ and $q^2\Lambda_q > 1$. The authors did not mention this different set of the parameters for the q -Hahn polynomials in [19]. However it is given in [20, page 76].

In Figure 7A, the only suitable interval is $(q^{-1}a_1, q^{-1}b_1)$ which coincides with the 1st case in Theorem 2.3. In fact, $\rho(qx, q)/\rho(x, q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$, then it follows

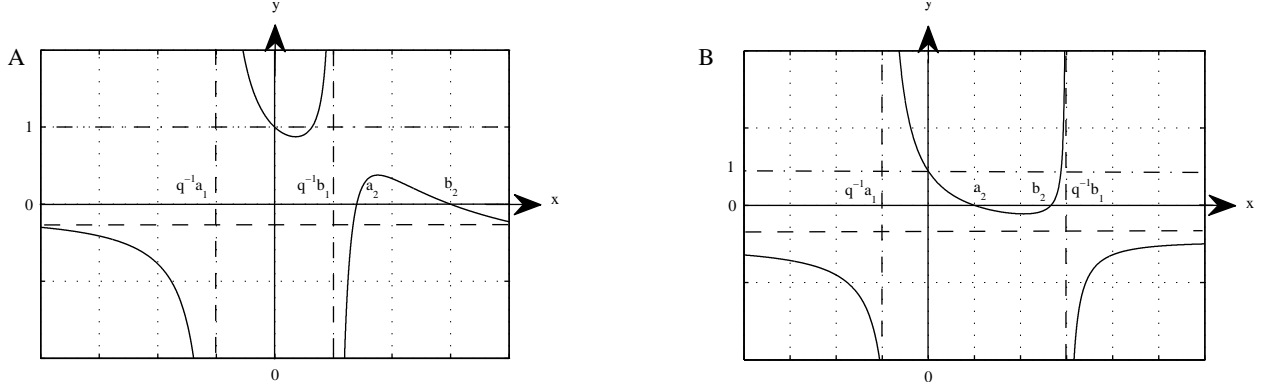


Figure 7: The graph of $f(x, q)$ in **Case 3(a)** with $\Lambda_q < 0$. In A, the zeros are in order $q^{-1}a_1 < 0 < q^{-1}b_1 < a_2 < b_2$, and in B, $q^{-1}a_1 < 0 < a_2 < b_2 < q^{-1}b_1$.

that ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$ with $\rho \rightarrow 0$ as $x \rightarrow q^{-1}a_1^+$ and $x \rightarrow q^{-1}b_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. Notice that the BCs (2.5) hold at $x = a_1$ and $x = b_1$. Then, there exists an OPS w.r.t. a ρ supported on the set of points $\{q^k a_1\}_{k \in \mathbb{N}_0} \cup \{q^k b_1\}_{k \in \mathbb{N}_0}$. Thus, we have the following result.

Theorem 4.5 *Consider the case $a_1 < 0 < b_1 < a_2 \leq b_2$, and $q^2 \Lambda_q < 0$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (4.4) supported on $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^k b\}_{k \in \mathbb{N}_0}$ (see (2.11) of Theorem 2.3-1).*

An example of this family is again the big q -Jacobi polynomials which are orthogonal on the set $\{cq, cq^2, cq^3, \dots\} \cup \{\dots, aq^3, aq^2, aq\}$. They satisfy the q -EHT with the coefficients in (4.5) where $a_1 = cq$, $b_1 = aq$, $a_2 = b^{-1}c$ and $b_2 = 1$. This case corresponds to the case VIIa1 in Chapter 10 of [19, pages 292 and 318]. However, notice that the conditions, $a_1 < 0 < b_1 < a_2 \leq b_2$ and $q^2 \Lambda_q < 0$, lead to the new constraints $c < 0$, $b < 0$, $abc^{-1}q \leq 1$ and $0 < a < q^{-1}$, which give a larger set of parameters for the orthogonality of the big q -Jacobi polynomials than the one reported in [19, page 319].

In Figure 7B, the only possible interval is $[b_2, q^{-1}b_1)$ which corresponds to the 3th case in Theorem 2.3. In this case $\rho(qx, q)/\rho(x, q) = 1$ at $b_2 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$, then it follows that ρ is increasing on $[b_2, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qb_2, q) = 0$ and $\rho(x, q) \rightarrow 0$ as $x \rightarrow q^{-1}b_1^-$ since $\rho(qb_2, q)/\rho(b_2, q) = 0$ and $\rho(qx, q)/\rho(x, q) \rightarrow \infty$ as $x \rightarrow q^{-1}b_1^-$. Thus there is a suitable ρ supported on the set of points $\{q^{-k}b_2\}_{k \in \mathbb{N}_0}$ where $q^{-N-1}b_2 = q^{-1}b_1$ (see Theorem 2.3-3). Hence we state the following theorem.

Theorem 4.6 *Consider the case where $a_1 < 0 < a_2 \leq b_2 < b_1$ and $q^2 \Lambda_q < 0$. Let $a = b_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x, q)$ and $\sigma_1(qx, q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (see Eq. 2 in Table 1)*

$$\rho(x, q) = |x|^t \frac{(\frac{qa}{x}, b^{-1}x; q)_\infty}{(\frac{a_1}{x}, a_2^{-1}x; q)_\infty}, \quad q^t = \frac{q^{-3}\sigma_2''(0, q)a_2}{\sigma_1''(0, q)b}$$

supported on the set of points $\{q^{-k}a\}_{k \in \mathbb{N}_0}$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-3).

A typical example of this family is again the q -Hahn polynomials orthogonal on $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$. They satisfy (1.1) and (1.3) given by (4.7), being $a_1 = \alpha q$, $b_1 = q^{-N}$, $a_2 = \beta^{-1}q^{-N-1}$ and $b_2 = 1$. The conditions $a_1 < 0 < a_2 \leq b_2 < b_1$ and $q^2 \Lambda_q < 0$ lead to the orthogonality relation for the q -Hahn polynomials that is valid in a larger set of the parameters, $\alpha < 0$ and $\beta \geq q^{-N-1}$. This new parameter set is not mentioned in [19].

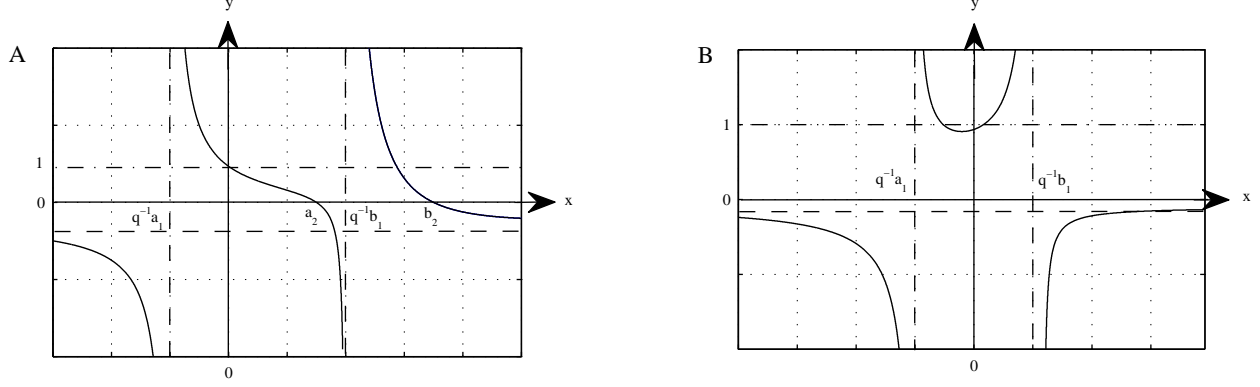


Figure 8: The graph of $f(x, q)$ in **Case 3** with $\Lambda_q < 0$. In A, we have **Case 3(a)** with $q^{-1}a_1 < 0 < a_2 < q^{-1}b_1 < b_2$. In B, we have **Case 3(c)** with $q^{-1}a_1 < 0 < q^{-1}b_1$ and $a_2, b_2 \in \mathbb{C}$.

We could not find an OPS in case of Figure 8A. In Figure 8B, the interval $(q^{-1}a_1, q^{-1}b_1)$ coincides with the 1st case in Theorem 2.3. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$ with $\rho(x, q) \rightarrow 0$ as $x \rightarrow q^{-1}a_1^+$ and $x \rightarrow q^{-1}b_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. Then there is a suitable ρ supported on the set $\{q^k a_1\}_{k \in \mathbb{N}_0} \cup \{q^k b_1\}_{k \in \mathbb{N}_0}$. Therefore, the following theorem holds:

Theorem 4.7 Consider the case $a_1 < 0 < b_1$, $a_2, b_2 \in \mathbb{C}$ and $q^2\Lambda_q < 0$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (4.4), supported on $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^k b\}_{k \in \mathbb{N}_0}$, (see (2.11) of Theorem 2.3-1) with

$$d_n^2 = (b_1 - a_1)(1 - q)q^{n(n-1)/2}(-a_1 b_1)^n \frac{(q, q^{-1}a_2^{-1}b_2^{-1}a_1 b_1; q)_n}{(q^{-1}a_2^{-1}b_2^{-1}a_1 b_1, a_2^{-1}b_2^{-1}a_1 b_1; q)_{2n}} \\ \times (a_2^{-1}a_1, a_2^{-1}b_1, b_2^{-1}a_1, b_2^{-1}b_1; q)_n \frac{(q, qb_1 a_1^{-1}, qa_1 b_1^{-1}, a_2^{-1}b_2^{-1}a_1 b_1; q)_\infty}{(a_2^{-1}a_1, a_2^{-1}b_1, b_2^{-1}a_1, b_2^{-1}b_1; q)_\infty}$$

where $q^2\Lambda_q = a_1 b_1 a_2^{-1} b_2^{-1}$, $a_2 = i\alpha$, $b_2 = \overline{a_2} = -i\alpha$, $\alpha \in \mathbb{R}$.

This case is included in the case VIIa1 in Chapter 10 of [19, pages 292 and 318] (with $\gamma_2 = \overline{\gamma_1}$) but it is not mentioned there. In fact, this case is similar to the big q -Jacobi polynomials studied in (**Cases 1** in Figure 3A and **Case 3(a)** in Figure 7A). The difference is that the roots $a_2(q)$ and $b_2(q)$ are complex.

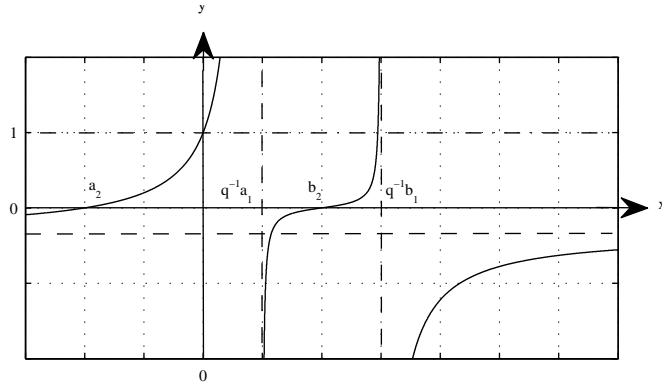


Figure 9: The graph of $f(x, q)$ in **Case 4** with $\Lambda_q < 0$. The zeros are in order $a_2 < 0 < q^{-1}a_1 < b_2 < q^{-1}b_1$.

In Figure 9, the only possible interval is $[b_2, q^{-1}b_1)$ which corresponds to the one described in Theorem 2.3-3. A similar analysis shows that there exists a q -weight function defined on the interval $[b_2, q^{-1}b_1)$ supported at the points $q^{-k}b_2$ for $k = 0, 1, \dots, N$ where $q^{-N-1}b_2 = q^{-1}b_1$ which lead to the following theorem:

Theorem 4.8 *Consider the case where $a_2 < 0 < a_1 < b_2 < b_1$ and $q^2\Lambda_q < 0$. Let $a = b_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x, q)$ and $\sigma_1(qx, q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (4.6) in Theorem (4.3) supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-3).*

An example of this family is again the q -Hahn polynomials orthogonal on $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$. They satisfy (1.1) and (1.3) with the coefficients (4.7) where $a_1 = \alpha q$, $b_1 = q^{-N}$, $a_2 = \beta^{-1}q^{-N-1}$ and $b_2 = 1$. The conditions $a_2 < 0 < a_1 < b_2 < b_1$ and $q^2\Lambda_q < 0$ lead to another new constraints $0 < \alpha < q^{-1}$ and $\beta < 0$ on the parameters of the q -Hahn polynomials which extend the orthogonality relation for the q -Hahn polynomials and it has been not reported in [19].

For completeness, we have also examined the cases listed below for which an OPS fails to exist.

Case 2(a) with $0 < a_2 < q^{-1}a_1 < b_2 < q^{-1}b_1$ and $\Lambda_q > 1$.

Case 2(a) with $0 < a_2 < q^{-1}a_1 < q^{-1}b_1 < b_2$ and $\Lambda_q > 1$.

Case 2(a) with $0 < q^{-1}a_1 < a_2 < b_2 < q^{-1}b_1$ and $\Lambda_q > 1$.

Case 2(a) with $a_2 < b_2 < 0 < q^{-1}a_1 < q^{-1}b_1$ and $\Lambda_q > 1$.

Case 2(a) with $0 < q^{-1}a_1 < a_2 < b_2 < q^{-1}b_1$ and $0 < \Lambda_q < 1$.

Case 2(a) with $0 < a_2 < q^{-1}a_1 < q^{-1}b_1 < b_2$ and $0 < \Lambda_q < 1$.

Case 2(a) with $0 < q^{-1}a_1 < q^{-1}b_1 < a_2 < b_2$ and $0 < \Lambda_q < 1$.

Case 2(a) with $a_2 < b_2 < 0 < q^{-1}a_1 < q^{-1}b_1$ and $0 < \Lambda_q < 1$.

Case 4 with $a_2 < 0 < q^{-1}a_1 < q^{-1}b_1 < b_2$ and $\Lambda_q < 0$.

Case 4 with $a_2 < 0 < b_2 < q^{-1}a_1 < q^{-1}b_1$ and $\Lambda_q < 0$.

4.2 The zero case

We make a similar analysis here with the same notations. Let the coefficients σ_1 and σ_2 be quadratic polynomials in x such that $\sigma_1(0, q) = \sigma_2(0, q) = 0$. If σ_1 is written as $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)x[x - a_1(q)]$ then, from (1.4) $\sigma_2(x, q) = \frac{1}{2}\sigma_2''(0, q)x^2 + \sigma_2'(0, q)x$ where

$$\frac{1}{2}\sigma_2''(0, q) = q \left[\frac{1}{2}\sigma_1''(0, q) + (1 - q^{-1})\tau'(0, q) \right] \neq 0 \quad \text{and} \quad \sigma_2'(0, q) = q(1 - q^{-1})\tau(0, q) - \frac{1}{2}\sigma_1''(0, q)a_1(q).$$

Then it follows from (1.5) that

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \left[1 + \frac{(1 - q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] \frac{[x - a_2(q)]}{q[qx - a_1(q)]}, \quad x \neq 0 \quad (4.9)$$

provided that $\left[1 + \frac{(1 - q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] a_2(q) = \left[a_1(q) - \frac{(1 - q^{-1})\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right]$. Let us point out that Λ_q defined in (4.2) is also horizontal asymptote of $f(x, q)$ in (4.9). Moreover, f intersects the y -axis at the point

$$y := y_0 = q^{-1} \left[1 - \frac{(1 - q^{-1})}{a_1(q)} \frac{\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right].$$

In the zero cases notice that one of the boundary of (a, b) interval could be zero. Therefore for such a case we need to know the behaviour of ρ at the origin.

Lemma 4.9 If $0 < y_0 < 1$, then $\rho(z, q) \rightarrow 0$ as $z \rightarrow 0$. Otherwise it diverges to $\mp\infty$.

Proof From (2.10), we write $\rho(q^k x, q) = \rho(x, q) \prod_{i=0}^{k-1} \frac{q^{-1}\sigma_2(q^i x, q)}{\sigma_1(q^{i+1} x, q)}$ from which

$$\rho(q^k x, q) = q^{-k} \left[1 - \frac{(1-q^{-1})}{a_1(q)} \frac{\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right]^k \frac{(x/a_2(q); q)_k}{(qx/a_1(q); q)_k} \rho(x, q) \quad (4.10)$$

is obtained. Taking $k \rightarrow \infty$ the result follows. ■

In a similar fashion, we introduce the two additional cases which include all possibilities.

Case 5. $\Lambda_q > 0$ with (a) $0 < y_0 < 1$ or (b) $y_0 > 1$ or (c) $y_0 < 0$.

Case 6. $\Lambda_q < 0$ with (a) $0 < y_0 < 1$ or (b) $y_0 > 1$ or (c) $y_0 < 0$.

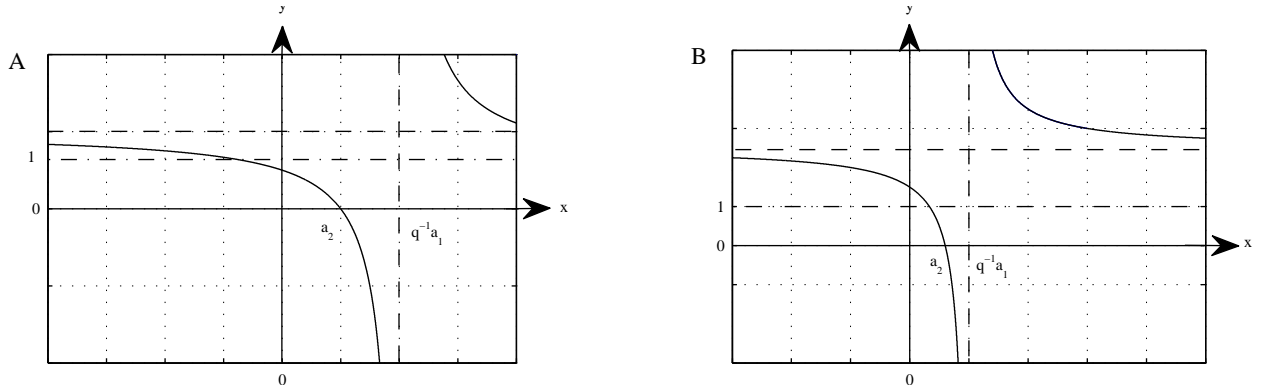


Figure 10: The graph of $f(x, q)$ in **Case 5** with $\Lambda_q > 1$ and $0 < a_2 < q^{-1}a_1$. In A, we have **Case 5(a)**. In B, we have **Case 5(b)**.

In Figure 10A, we consider all possible intervals in which we can have a suitable q -weight function ρ . By the positivity of ρ , the interval $(a_2, q^{-1}a_1)$ is not suitable. The other intervals $(0, a_2)$ and $(q^{-1}a_1, \infty)$ are both eliminated due to the **PIV** and **PV**, respectively. The interval $(-\infty, 0)$ is the one described in Theorem 2.3-6 by symmetry. Since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) < 0$, ρ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, 0)$. Moreover, since $0 < y_0 < 1$ $\rho \rightarrow 0$ as $x \rightarrow 0^-$ according to Lemma 4.9. On the other hand, since $\rho(qx, q)/\rho(x, q)$ has a finite limit as $x \rightarrow -\infty$, we may have $\rho \rightarrow 0$ as $x \rightarrow -\infty$, but we should check that $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow -\infty$ by using the extended q -Pearson equation (4.3). However, the graph of g (4.3) looks like the one represented in Figure 10A together with the property that $g(x, q) < 1$ on $(-\infty, 0)$ for large k which leads to that $\sigma_1(x, q)\rho(x, q)x^k$ is decreasing on $(-\infty, 0)$ with $\sigma_1(x, q)\rho(x, q)x^k \not\rightarrow 0$ as $x \rightarrow -\infty$. Therefore, this case does not lead to any suitable ρ and, therefore, OPS. The same result is valid for the case in Figure 10B.

In Figure 11A, the only suitable interval is $(0, q^{-1}a_1)$ which coincides with the 2nd case in Theorem 2.3. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $0 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}a_1$, then ρ is increasing on $(0, x_0)$ with $\rho \rightarrow 0$ as $x \rightarrow 0^+$ since $0 < y_0 < 1$ and decreasing on $(x_0, q^{-1}a_1)$ with $\rho(x, q) \rightarrow 0$ as $x \rightarrow q^{-1}a_1^-$. Thus there is a q -weight function supported at the points $q^k a_1$ for $k \in \mathbb{N}_0$. Hence, the resulting OPS is introduced in Theorem 4.10. But, the case in Figure 11B does not yield any OPS.

Theorem 4.10 Consider the case where $0 < a_1 < a_2$, $0 < qy_0 < 1$ and $0 < q^2\Lambda_q < 1$. Let $a = 0$ and $b = a_1(q)$ be the zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see Eq. 3 in Table 1)

$$\rho(x, q) = |x|^\alpha \frac{(qb^{-1}x; q)_\infty}{(a_2^{-1}x; q)_\infty}, \quad q^\alpha = \frac{q^{-2}\sigma_2''(0, q)a_2}{\sigma_1''(0, q)b}$$

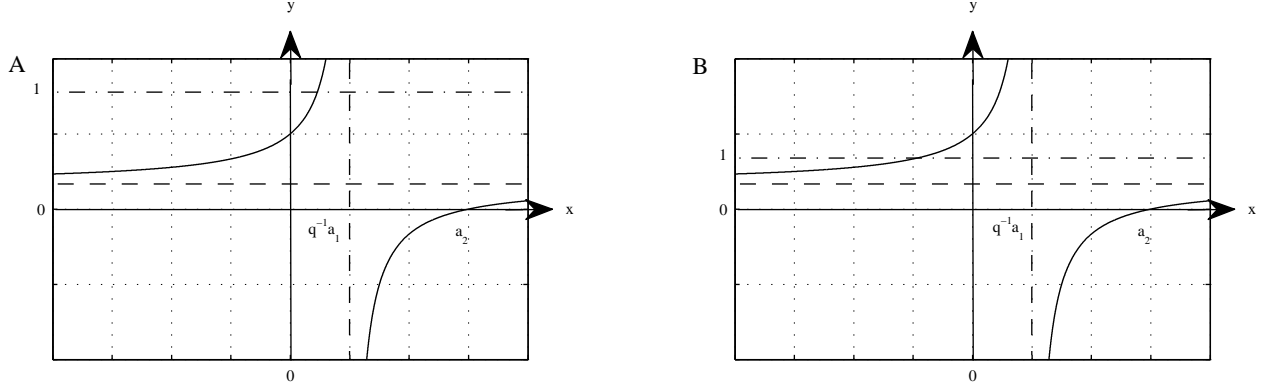


Figure 11: The graph of $f(x, q)$ in **Case 5** with $0 < \Lambda_q < 1$ and $0 < q^{-1}a_1 < a_2$. In A, we have **Case 5(a)**. In B, we have **Case 5(b)**.

supported on $\{q^k b\}_{k \in \mathbb{N}_0}$ (see (2.12) of Theorem 2.3-2).

The OPS in Theorem 4.10 corresponds to the case IVa3 in Chapter 10 of [19, pages 277 and 311]. In fact, a typical example of this family is the little q -Jacobi polynomials $P_n(x; a, b|q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-2}x(x - a_1), \quad \sigma_2(x, q) = abqx(x - a_2),$$

$$\tau(x, q) = \frac{1 - abq^2}{(1 - q)q}x + \frac{aq - 1}{(1 - q)q} \quad \text{and} \quad \lambda_n(q) = -q^{-n}[n]_q \frac{1 - abq^{n+1}}{1 - q} \quad (4.11)$$

where $a_1 = 1$ and $a_2 = b^{-1}q^{-1}$. The conditions $0 < q^2\Lambda_q < 1$, $0 < qy_0 < 1$ and $0 < a_1 < a_2$ yield the restrictions $0 < a < q^{-1}$ and $0 < b < q^{-1}$ on the parameters of $P_n(x; a, b|q)$ with orthogonality on $\{\dots, q^2, q, 1\}$ in the sense (2.12) where

$$d_n^2 = a^n q^{n^2} (1 - q) \frac{(q, abq; q)_n}{(abq, abq^2; q)_{2n}} (aq, bq; q)_n \frac{(q, abq^2; q)_\infty}{(aq, bq; q)_\infty}. \quad (4.12)$$

In the literature, this relation can be found as an infinite sum [19, Page 312].

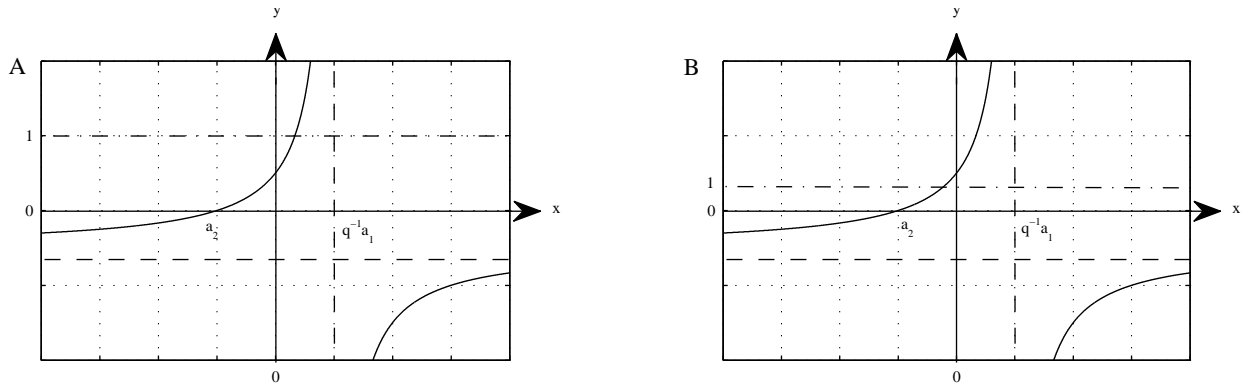


Figure 12: The graph of $f(x, q)$ in **Case 6** with $\Lambda_q < 0$ and $a_2 < 0 < q^{-1}a_1$. In A, we have **Case 6(a)**. In B, we have **Case 6(b)**.

In Figure 12A, the only interval is $(0, q^{-1}a_1)$ which corresponds to the interval described in Theorem 2.3-2. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $0 < x_0 = -\tau'(0, q)/\tau(0, q) < q^{-1}a_1$ then ρ is increasing on $(0, x_0)$ and decreasing on $(x_0, q^{-1}a_1)$. Furthermore, since $0 < y_0 < 1$, $\rho \rightarrow 0$ as $x \rightarrow 0^+$ according to

Lemma 4.9 and $\rho \rightarrow 0$ as $x \rightarrow q^{-1}a_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow +\infty$ as $x \rightarrow q^{-1}a_1^-$. Then, there exists a suitable ρ on $(0, a_1]$ supported at the points $q^k a_1$ for $k \in \mathbb{N}_0$ and, therefore, an OPS exists which is stated in the following theorem. However, the case analysed in Figure 12B does not give any OPS.

Theorem 4.11 *Consider the case where $a_2 < 0 < a_1$, $0 < qy_0 < 1$ and $q^2\Lambda_q < 0$. Let $a = 0$ and $b = a_1(q)$ be zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on $(a, b]$ w.r.t. the weight function in Theorem 4.10, supported on $\{q^k b\}_{k \in \mathbb{N}_0}$ (see (2.12) of Theorem 2.3-2).*

The OPS in Theorem 4.11 coincides with the case IVa4 in Chapter 10 of [19, pages 278 and 312]. An example of this family is again the little q -Jacobi polynomials with orthogonality on the set of points $\{\dots, q^2, q, 1\}$. They satisfy the q -EHT with the coefficients in (4.11) where $a_1 = 1$ and $a_2 = b^{-1}q^{-1}$. These polynomials have the same orthogonality property with the same d_n^2 as in (4.12) but the constraints $0 < a < q^{-1}$ and $b < 0$ on the parameters are different due to the conditions. This extends the orthogonality relation of the little q -Jacobi polynomials for $0 < a < q^{-1}$ and $0 < b < q^{-1}$ to a larger set of the parameters $0 < a < q^{-1}$ and $b < 0$. Notice that combining this with the previous Case 5(a) one can obtain the orthogonality relation of the little q -Jacobi polynomials for $0 < aq < 1, bq < 1$.

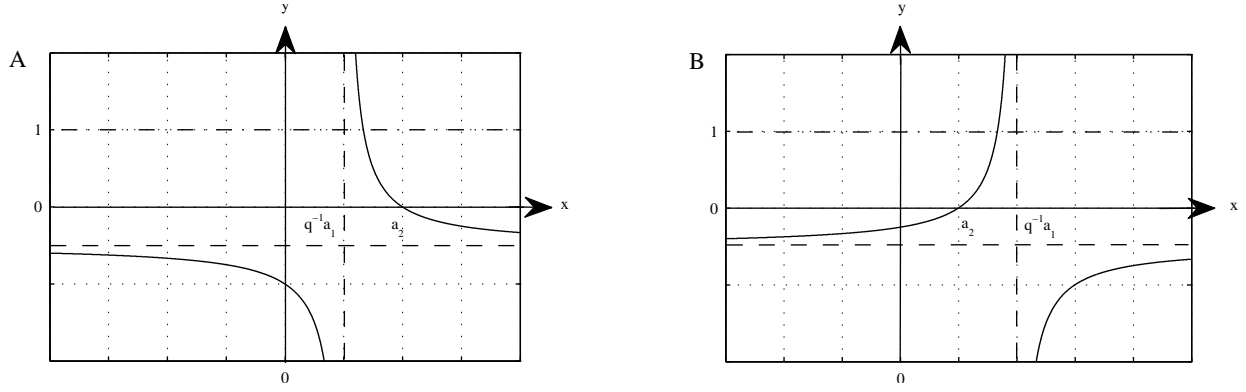


Figure 13: The graph of $f(x, q)$ in **Case 6(c)** with $\Lambda_q < 0$. In A, the zeros are in order $0 < q^{-1}a_1 < a_2$. In B, the zeros are in order $0 < a_2 < q^{-1}a_1$.

The case in Figure 13A does not yield any OPS. On the other hand, in Figure 13B, the only possible interval is $[a_2, q^{-1}a_1]$ which is 3th case in Theorem 2.3. Note that $\rho(qx, q)/\rho(x, q) = 1$ at $a_2 < x_0 = -\tau'(0, q)/\tau(0, q) < q^{-1}a_1$ and that ρ is increasing on (a_2, x_0) and decreasing on $(x_0, q^{-1}a_1)$. Furthermore, $\rho(qa_2, q) = 0$ since $\rho(qa_2, q)/\rho(a_2, q) = 0$ and $\rho \rightarrow 0$ as $x \rightarrow q^{-1}a_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$ as $x \rightarrow q^{-1}a_1^-$. Then there is an OPS on $\{q^{-k}a_2\}_{k=0}^N$ where $q^{-N-1}a_2 = q^{-1}a_1$. Therefore, we have the following theorem.

Theorem 4.12 *Consider the case where $0 < a_2 < a_1$, $qy_0 < 0$ and $q^2\Lambda_q < 0$. Let $a = a_2(q)$ and $b = q^{-1}a_1(q)$ be zeros of $\sigma_2(x, q)$ and $\sigma_1(qx, q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (see Eq. 4 in Table 1)*

$$\rho(x, q) = |x|^\alpha \sqrt{x^{\log_q x - 1}} \left(\frac{qa}{x}, b^{-1}x; q \right)_\infty, \quad q^\alpha = -\frac{q^{-3}\sigma_2''(0, q)}{\sigma_1''(0, q)b}$$

supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-3).

An example of this family is the q -Kravchuk polynomials $K_n(x; p, N; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-2}x(x - a_1), \quad \sigma_2(x, q) = -px(x - a_2),$$

$$\tau(x, q) = \frac{1 + pq}{(1 - q)q}x - \frac{p + q^{-N-1}}{1 - q} \quad \text{and} \quad \lambda_n(q) = -q^{-n}[n]_q \frac{1 + pq^n}{1 - q}$$

where $a_1 = q^{-N}$ and $a_2 = 1$. The conditions $q^2\Lambda_q < 0$, $qy_0 < 0$ and $0 < a_2 < a_1$ lead to the condition $p > 0$ on the parameter of $K_n(x; p, N; q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$ in the sense (2.13) where

$$d_n^2 = (q^{-1} - 1)p^{-N}q^{-(2^{N+1})}(-q^{-N}p)^n q^{n^2} \frac{1 + p}{1 + pq^{2n}}(-pq; q)_N(q, q^{N+1}; q)_\infty \frac{(q, -pq^{N+1}; q)_n}{(-p, q^{-N}; q)_n}. \quad (4.13)$$

In the literature, this relation is usually written as a finite sum [20, Page 98]. This case is not mentioned in [19] for $0 < q < 1$. However the q -Kravchuk polynomials with this set of parameters are described in [20, page 98].

In the following independent cases we fail to define an OPS.

Case 5(a) with $0 < \Lambda_q < 1$ and $0 < a_2 < q^{-1}a_1$.

Case 5(b) with $\Lambda_q > 1$ and $0 < q^{-1}a_1 < a_2$.

Case 5(c) with $\Lambda_q > 1$ and $q^{-1}a_1 < 0 < a_2$.

Case 5(c) with $0 < \Lambda_q < 1$ and $a_2 < 0 < q^{-1}a_1$.

5 The orthogonality of the q -polynomials: other cases

This section includes the main analysis of the other families by taking into account the rational function on the r.h.s. of the q -Pearson equation (1.5) along the same lines with the \emptyset -Jacobi/Jacobi and 0-Jacobi/Jacobi cases handled in the previous section.

5.1 q -Classical \emptyset -Jacobi/Laguerre Polynomials

Let the coefficients σ_2 and σ_1 be quadratic and linear polynomials in x , respectively, such that $\sigma_1(0, q)\sigma_2(0, q) \neq 0$. If σ_1 is written in terms of its root, i.e., $\sigma_1(x, q) = \sigma'_1(0, q)[x - a_1(q)]$, $a_1(q) = -\frac{\sigma_1(0, q)}{\sigma'_1(0, q)}$ then from (1.4)

$$\sigma_2(x, q) = (q - 1)\tau'(0, q)x^2 + [q\sigma'_1(0, q) + (q - 1)\tau(0, q)]x - q\sigma'_1(0, q)a_1(q)$$

where $\tau'(0, q) \neq 0$ by hypothesis. Then the q -Pearson equation (1.5) takes the form

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)} = \frac{(1 - q^{-1})\frac{\tau'(0, q)}{\sigma'_1(0, q)}[x - a_2(q)][x - b_2(q)]}{qx - a_1(q)} \quad (5.1)$$

provided that the discriminant denoted by Δ_q ,

$$\Delta_q := \left[1 + \frac{(1 - q^{-1})\tau(0, q)}{\sigma'_1(0, q)}\right]^2 + 4a_1(q)(1 - q^{-1})\frac{\tau'(0, q)}{\sigma'_1(0, q)}$$

of the quadratic polynomial in the nominator of f in (5.1) is non-zero. Note that here $x = a_2$ and $x = b_2$ are roots of f which are constant multiplies of the roots of σ_2 . Moreover, $x = q^{-1}a_1$ is the vertical asymptote of f and $y = 1$ is its y -intercept since $\sigma_2(0, q) = q\sigma_1(0, q)$. On the other hand, the locations of the zeros of f are introduced by the following straightforward lemma.

Lemma 5.1 Let $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} \neq 0$. Then, we have the following cases for the roots of the equation $f(x, q) = 0$.

Case 1. If Λ_q and $a_1(q)$ have opposite signs, then there are two real distinct roots with opposite signs.

Case 2. If Λ_q and $a_1(q)$ have same signs, then there exist three possibilities

- (a) if $\Delta_q > 0$, f has two real roots with same signs
- (b) if $\Delta_q = 0$, f has a double root
- (c) if $\Delta_q < 0$, f has a pair of complex conjugate roots.

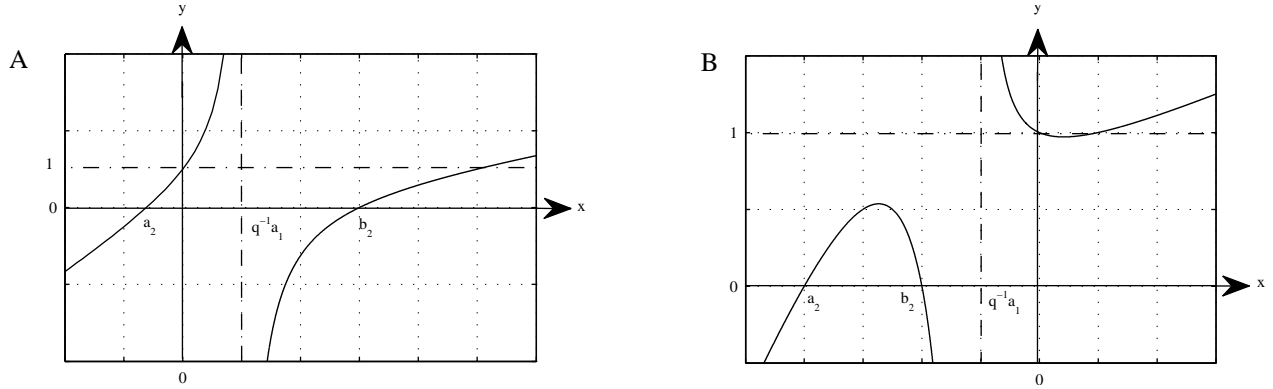


Figure 14: The graph of $f(x, q)$. In A, we have **Case 1** with $\Lambda_q < 0$ and $a_2 < 0 < q^{-1}a_1 < b_2$, and in B, we have **Case 2(a)** with $\Lambda_q < 0$ and $a_2 < b_2 < q^{-1}a_1 < 0$.

In Figure 14A, we first start with positivity condition of q -weight function which allows us to exclude the intervals $(-\infty, a_2)$ and $(q^{-1}a_1, b_2)$. Moreover, due to **PIII** $(a_2, q^{-1}a_1)$ can not be used. On the other hand, the interval (b_2, ∞) coincides with the 5th case of Theorem 2.3. Notice that since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > b_2$, ρ is decreasing on (x_0, ∞) . Moreover, Since $\rho(qx, q)/\rho(x, q)$ has an infinite limit as $x \rightarrow +\infty$, we have $\rho \rightarrow 0$ as $x \rightarrow \infty$. However, since it is infinite interval, we should check that $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ by using extended q -Pearson equation (4.3). The graph of the function g defined in (4.3) looks like the one for f . Then the analysis of the extended q -Pearson equation leads to $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$. Therefore, there exists a suitable ρ supported on the set of points $\{q^{-k}b_2\}_{k \in \mathbb{N}_0}$. Thus, we have the following theorem.

Theorem 5.2 Let $a_2 < 0 < a_1 < b_2$ and $\Lambda_q < 0$. Let $a = b_2(q)$ be the zero of $\sigma_2(x, q)$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 2nd expression of the \emptyset -Jacobi/Laguerre case in Table 2)

$$\rho(x, q) = |x|^\alpha \sqrt{x^{\log_q x - 1}} \frac{(qa_2/x, qa/x; q)_\infty}{(a_1/x; q)_\infty}, \quad q^\alpha = \frac{q^{-2\frac{1}{2}}\sigma_2''(0, q)}{\sigma_1'(0, q)} \quad (5.2)$$

supported on $\{q^{-k}a\}_{k \in \mathbb{N}_0}$ (see (2.15) of Theorem 2.3-5).

The OPS in Theorem 5.2 coincides with the case IIa2 in Chapter 11 of [19, pages 337 and 358]. An example of this family is the q -Meixner polynomials $M_n(x; b, c; q)$ satisfying the q -EHT with the coefficients

$$\begin{aligned} \sigma_1(x, q) &= cq^{-2}(x - a_1), \quad \sigma_2(x, q) = (x - a_2)(x - b_2), \\ \tau(x, q) &= -\frac{1}{1-q}x + \frac{cq^{-1} - bc + 1}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{[n]_q}{1-q} \end{aligned} \quad (5.3)$$

where $a_1 = bq$, $a_2 = -bc$ and $b_2 = 1$. The conditions $\Lambda_q < 0$ and $a_2 < 0 < a_1 < b_2$ give us the known restrictions $c > 0$ and $0 < b < q^{-1}$ on the parameters of $M_n(x; b, c; q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots\}$ in the sense (2.15) where

$$d_n^2 = (q^{-1} - 1)c^{2n}q^{-n(2n+1)}(q, -c^{-1}q, bq; q)_n \frac{(q, -c; q)_\infty}{(bq; q)_\infty}.$$

In the literature, this relation can be found as an infinite sum [19, page 360].

In Figure 14B, the only possible interval is $(q^{-1}a_1, \infty)$ which is the one identified in Theorem 2.3-4. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > q^{-1}a_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on (x_0, ∞) which leads to $\rho \rightarrow 0$ as $x \rightarrow \infty$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. But we still need to show $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ by using the extended q -Pearson equation (4.3). By applying the same procedure to the extended q -Pearson equation (4.3) whose graph looks like the one for f , we get $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$. Consequently, we have a suitable ρ on the interval $[a_1, \infty)$ supported on the set of points $\{a_1q^k\}_{k \in \mathbb{N}_0} \cup \{q^{\mp k}\}_{k \in \mathbb{N}_0}$.

Theorem 5.3 *Let $a_2 \leq b_2 < a_1 < 0$, $\Lambda_q < 0$. Let $a = a_1$ be a zero of $\sigma_1(x, q)$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 1st expression of the \emptyset -Jacobi/Laguerre case in Table 2)*

$$\rho(x, q) = \frac{(a^{-1}qx; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$$

supported on the set of points $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^{\pm k} \alpha\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ in the sense (2.14) of Theorem 2.3-4 with

$$d_n^2 = (1 - q)q^{-n(2n-1)}(a_2 b_2 a_1^{-1})^{2n}(q, a_2^{-1}a_1, b_2^{-1}a_1; q)_n \frac{(q, a_1, qa_1^{-1}, a_2^{-1}b_2^{-1}a_1, qa_2 b_2 a_1^{-1}; q)_\infty}{(a_2^{-1}a_1, b_2^{-1}a_1, a_2^{-1}, b_2^{-1}, qa_2, qb_2; q)_\infty}. \quad (5.4)$$

The OPS in Theorem 5.3 coincides with the case VIa2 in Chapter 10 of [19, pages 285 and 315].

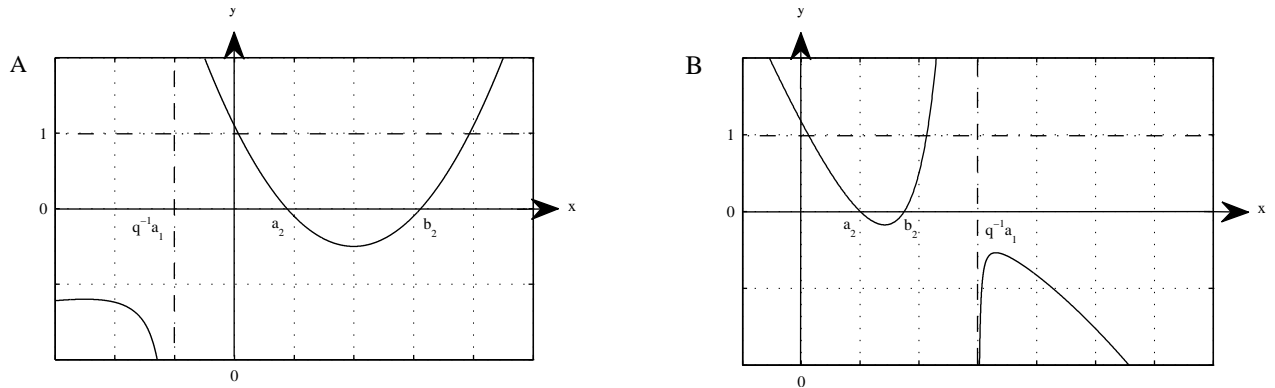


Figure 15: The graph of $f(x, q)$ in **Case 2(a)**. In A, we have $\Lambda_q < 0$ and $q^{-1}a_1 < 0 < a_2 < b_2$ and in B, $\Lambda_q > 0$ and $0 < a_2 < b_2 < q^{-1}a_1$.

In Figure 15A, the only possible interval is (b_2, ∞) . An analogous analysis as the one that has been done for the case in Figure 14A yields $\rho \rightarrow 0$ as $x \rightarrow \infty$. Moreover, since from (4.3) $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ for $k \in \mathbb{N}_0$, then there exists a q -weight function on $[b_2, \infty)$ supported at the points $q^{-k}b_2$ for $k \in \mathbb{N}_0$. Thus we have the following result.

Theorem 5.4 *Let $a_1 < 0 < a_2 \leq b_2$ and $\Lambda_q < 0$. Let $a = b_2$ be the zero of $\sigma_2(x, q)$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (5.2) (see Theorem 5.2) supported on $\{q^{-k}a\}_{k \in \mathbb{N}_0}$ (see (2.15) of Theorem 2.3-5).*

A typical example of this family is again the q -Meixner polynomials orthogonal on $\{1, q^{-1}, q^{-2}, \dots\}$. They satisfy the q -EHT with the coefficients (5.3) where $a_1 = bq$, $a_2 = -bc$ and $b_2 = 1$. This set of q -Meixner polynomials corresponds to the case IIa2 in Chapter 11 of [19, pages 337 and 358] and their orthogonality relation is valid in a larger set of parameters. In fact the conditions $a_1 < 0 < a_2 \leq b_2$ and $\Lambda_q < 0$ yield $c > 0$, $b < 0$ and $0 < -bc \leq 1$. This was not reported in [19].

In Figure 15B, the only possible interval is $[b_2, q^{-1}a_1]$ which coincides with 3th case of Theorem 2.3. In fact, $\rho(qx, q)/\rho(x, q) = 1$ at $b_2 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}a_1$, then ρ is increasing on (b_2, x_0) and decreasing on $(x_0, q^{-1}a_1)$. Moreover, $\rho(qb_2, q) = 0$ and $\rho(q^{-1}a_1, q) = 0$ since $\rho(qb_2, q)/\rho(b_2, q) = 0$ and $\rho(qx, q)/\rho(x, q) \rightarrow \infty$ as $x \rightarrow q^{-1}a_1^-$. Therefore, there is an OPS on $[b_2, q^{-1}a_1]$ w.r.t. a weight function supported on the set of points $\{q^{-k}b_2\}_{k=0}^N$ where $q^{-N-1}b_2 = q^{-1}a_1$.

Theorem 5.5 *Let $0 < a_2 \leq b_2 < a_1$ and $\Lambda_q > 0$. Let $a = b_2$ be the zero of $\sigma_2(x, q)$ and $b = q^{-1}a_1$ of $\sigma_1(qx, q)$. Then there exists a finite family of polynomials $(P_n)_n$ orthogonal w.r.t. the weight function (see the 3th expression of \emptyset -Jacobi/Laguerre in Table 2)*

$$\rho(x, q) = |x|^\alpha x^{\log_q x} (b^{-1}x, qa_2/x, a/x; q)_\infty, \quad q^\alpha = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\sigma_1'(0, q)b}$$

supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-3).

The OPS in Theorem 5.5 coincides with the case IIb1 in Chapter 11 of [19, pages 337 and 361]. An example of this family is the quantum q -Kravchuk polynomials $K_m^{qtm}(x; p, N; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = -q^{-2}(x - a_1), \quad \sigma_2(x, q) = p(x - a_2)(x - b_2),$$

$$\tau(x, q) = -\frac{p}{1-q}x + \frac{p - q^{-1} + q^{-N-1}}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{p}{1-q}[n]_q$$

where $a_1 = q^{-N}$, $a_2 = p^{-1}q^{-N-1}$ and $b_2 = 1$. The conditions $\Lambda_q > 0$ and $0 < a_2 \leq b_2 < a_1$ give the constrain $p \geq q^{-N-1}$ on the parameter of $K_m^{qtm}(x; p, N; q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$ in the sense (2.13) where

$$d_n^2 = (q^{-1} - 1) \frac{1}{(p^{-1}q^{-N}; q)_N} p^{-2n} q^{-n(2n+1)} (q, pq, q^{-N}; q)_n (q, p^{-1}q^{-N}, q^{N+1}; q)_\infty.$$

In the literature, this relation is usually written as a finite sum [19, page 362].

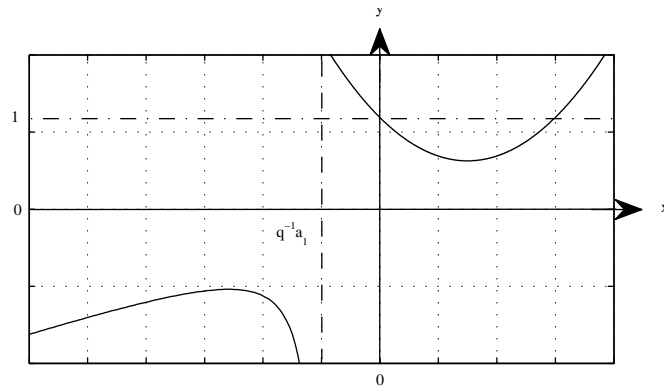


Figure 16: The graph of $f(x, q)$ in **Case 2(c)** with $\Lambda_q < 0$ and $a_1 < 0$, $a_2, b_2 \in \mathbb{C}$.

In Figure 16, $(q^{-1}a_1, \infty)$ is the only interval where f is positive. Notice that the graphs of f in the interval $(q^{-1}a_1, \infty)$ in Figures 16 and 14B have the same behaviour. Then, the analysis of Figure 14B is valid for this case and therefore there exists a suitable ρ on $(q^{-1}a_1, \infty)$. Thus, we have the following theorem.

Theorem 5.6 Let $a_1 < 0$, $a_2, b_2 \in \mathbb{C}$ and $\Lambda_q < 0$. Let $a = a_1$ be the zero of $\sigma_1(x, q)$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function, given in Theorem 5.3, supported on the set of points $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^{\pm k} \alpha\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ in the sense (2.14) of Theorem 2.3-4 with d_n^2 defined by (5.4)

The OPS in Theorem 5.6 coincides with the case VIa1 in Chapter 10 of [19, pages 285 and 315]. The orthogonality relation of this OPS has the same form as in the previous Case 2(a) defined in Theorem 5.3 but now the zeros of σ_2 are complex.

For the two cases listed below the OPS fails to exist.

Case 1. $\Lambda_q > 0$ and $q^{-1}a_1(q) < a_2(q) < 0 < b_2(q)$ and **Case 2(a).** $\Lambda_q > 0$ and $0 < a_2(q) < q^{-1}a_1(q) < b_2(q)$.

5.2 q -Classical \emptyset -Jacobi/Hermite Polynomials

Let the coefficients σ_2 and σ_1 be quadratic and constant polynomials in x , respectively, such that $\sigma_1(0, q)\sigma_2(0, q) \neq 0$. If $\sigma_1(x, q) = \sigma_1(0, q) \neq 0$ then, from (1.4),

$$\sigma_2(x, q) = q [\sigma_1(x, q) + (1 - q^{-1})x\tau(x, q)] = (q - 1)\tau'(0, q)x^2 + (q - 1)\tau(0, q)x + q\sigma_1(0, q)$$

where $\tau'(0, q) \neq 0$ by hypothesis. Then the q -Pearson equation (1.5) takes the form

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)} = (1 - q^{-1})\frac{\tau'(0, q)}{\sigma_1(0, q)}[x - a_2(q)][x - b_2(q)] \quad (5.5)$$

provided that the discriminant denoted by Δ_q ,

$$\Delta_q := \left[(1 - q^{-1})\frac{\tau(0, q)}{\sigma_1(0, q)} \right]^2 - 4(1 - q^{-1})\frac{\tau'(0, q)}{\sigma_1(0, q)}$$

of f in (5.5) is non-zero. Notice that y -intercept of f is $y = 1$ since $\sigma_2(0, q) = q\sigma_1(0, q)$. Moreover, $x = a_2$ and $x = b_2$ indicate its zeros which are constant multiples of the roots of σ_2 . The following straightforward lemma allows us to determine the locations of the zeros of f .

Lemma 5.7 Let $\Lambda_q = \frac{\tau'(0, q)}{\sigma_1(0, q)} \neq 0$. Then we encounter the following cases for the roots of the equation $f(x, q) = 0$.

Case 1. If $\Lambda_q > 0$, f has two real distinct roots with opposite signs.

Case 2. If $\Lambda_q < 0$, there exist three possibilities

- (a) if $\Delta_q > 0$, f has two real roots with same signs
- (b) if $\Delta_q = 0$, f has a double root
- (c) if $\Delta_q < 0$, f has a pair of complex conjugate roots.

The next step is sketching roughly all graphs of f by taking into account all possible relative positions of the zeros of f in question. As a result of analysis of the graphs of f , we determine a suitable $\rho > 0$ satisfying the q -Pearson equation (1.5) with BCs (2.5), (2.7).

In Figure 17A, let us consider the possible intervals in which we can have a suitable weight function ρ which are defined by the zeros of the polynomials σ_1 and σ_2 . First of all, notice that since ρ should be a positive weight function and f is negative in the intervals $(-\infty, a_2)$ and (b_2, ∞) , they are not suitable. On the other hand, the interval (a_2, b_2) is also eliminated in which $\rho = 0$ due to **PII**. As a result, an OPS fails to exist.

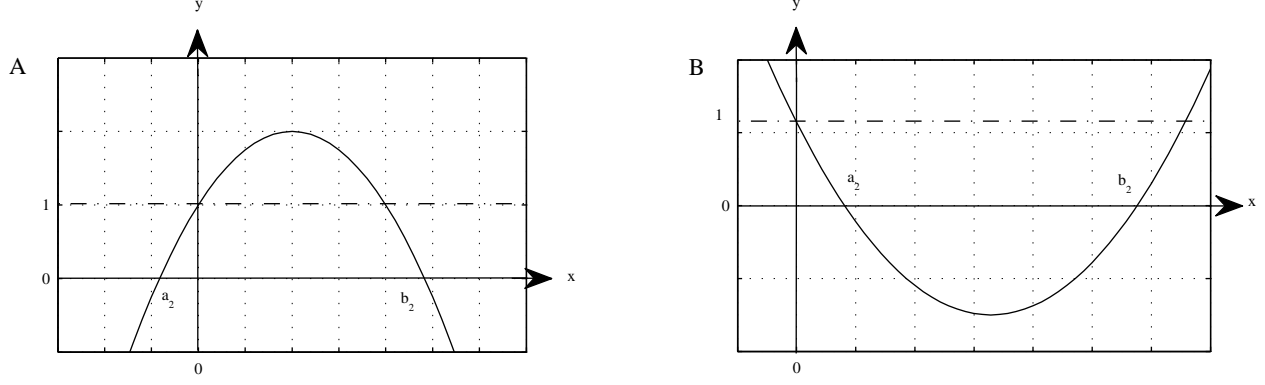


Figure 17: The graph of $f(x, q)$. In A, we have **Case 1** with $\Lambda_q > 0$ and $a_2 < 0 < b_2$, and in B, **Case 2(a)** with $\Lambda_q < 0$ and $0 < a_2 < b_2$.

Let us analyse the case in Figure 17B. The positivity of ρ implies that the interval (a_2, b_2) should be eliminated. On the other hand, $(-\infty, a_2)$ is not suitable since $\rho = 0$ in $(0, a_2)$ (this situation is similar to the one described in **PVI**). The interval (b_2, ∞) coincides with the 5th case of Theorem 2.3. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > b_2$, then ρ is decreasing on (x_0, ∞) . Since f has infinite limit as $x \rightarrow +\infty$, $\rho \rightarrow 0$ as $x \rightarrow \infty$. As a result, the typical shape of ρ is constructed in Figure 18 assuming a positive initial value of ρ in each subinterval.

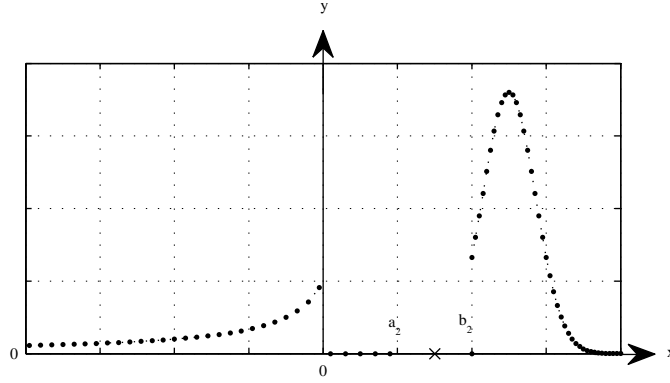


Figure 18: The graph of $\rho(x, q)$ associated with the case in Figure 17B.

However, it is not enough to assure that ρ satisfies the BC at $+\infty$. In fact, even if $\rho \rightarrow 0$ as $x \rightarrow \infty$ we should check that $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ by using the extended q -Pearson equation (4.3), which is represented in Figure 19 for some $0 < q < 1$, where k is large enough.

If we now provide a similar analysis for g in (4.3), we see from Figure 19 that, g has the same property with f . Therefore, $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$. That is, an OPS, to be stated in Theorem 5.8, exists on the supporting set of points $\{b_2 q^{-k}\}_{k \in \mathbb{N}_0}$.

Theorem 5.8 *Let $0 < a_2 \leq b_2$ and $\Lambda_q < 0$. Let $a = b_2(q)$ be a zero of $\sigma_2(x, q)$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see expression 2 for the \emptyset -Jacobi/Hermite case in Table 2)*

$$\rho(x, q) = |x|^\alpha x^{\log_q x - 1} (qa_2/x, qa/x; q)_\infty, \quad q^\alpha = \frac{q^{-1} \frac{1}{2} \sigma_2''(0, q)}{\sigma_1(0, q)}$$

supported on the set of points $\{q^{-k}a\}_{k \in \mathbb{N}_0}$ (see (2.15) of Theorem 2.3-5).

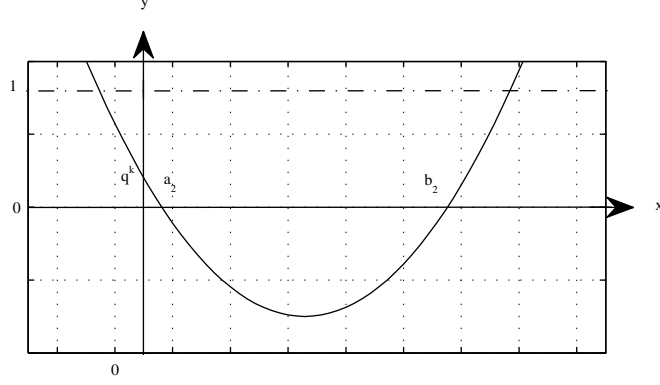


Figure 19: The graph of $g(x, q)$ corresponding to Figure 17B.

The OPS in Theorem 5.8 coincides with the case Ia1 in Chapter 11 of [19, pages 335 and 355-357]. In fact, a typical example of this family is the Al-Salam-Carlitz II polynomials $V_n^{(\alpha)}(x; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = aq^{-1}, \quad \sigma_2(x, q) = (x - a_2)(x - b_2),$$

$$\tau(x, q) = \frac{1}{q-1}x - \frac{1+a}{q-1} \quad \text{and} \quad \lambda_n(q) = \frac{1}{1-q}[n]_q$$

where $a_2 = a$, $b_2 = 1$. The conditions $\Lambda_q < 0$ and $0 < a_2 \leq b_2$ give the constrain $0 < a \leq 1$ on the parameter of $V_n^{(\alpha)}(x; q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots\}$ in the sense (2.15) where

$$d_n^2 = (q^{-1} - 1)a^n q^{-n^2} (q; q)_n (q; q)_\infty.$$

In the literature, this relation is usually written as an infinite sum [19, page 357].

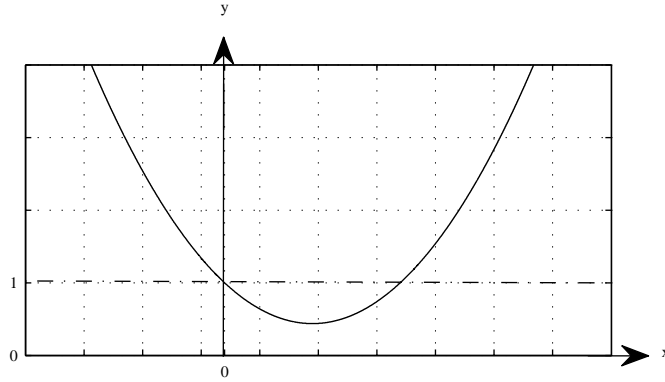


Figure 20: The graph of $f(x, q)$ in **Case2(c)** with $\Lambda_q < 0$ and $a_2(q), b_2(q) \in \mathbb{C}$.

In Figure 20, the only interval is $(-\infty, \infty)$ which corresponds to the 7th case of Theorem 2.3. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, then it follows that ρ is increasing on $(-\infty, x_0)$ and decreasing on (x_0, ∞) . Moreover, $\rho \rightarrow 0$ as $x \rightarrow \mp\infty$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. Then, an OPS can exist on $\{\pm q^{\mp k}\}_{k \in \mathbb{N}_0}$. But we should analyse the extended q -Pearson equation (4.3) to check $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \mp\infty$ which leads to similar figure as Figure 20. Then ρ and $\sigma_1(x, q)\rho(x, q)x^k$ have same property that $q\sigma_1(x, q)\rho(x, q)x^k = \sigma_2(q^{-1}x, q)\rho(q^{-1}x, q)x^k \rightarrow 0$ as $x \rightarrow \mp\infty$ for $k \in \mathbb{N}_0$. Thus we can find a suitable ρ supported on the set of points $\{\pm q^{\mp k}\}_{k \in \mathbb{N}_0}$. Therefore, we have the following theorem.

Theorem 5.9 Let $\Lambda_q < 0$ and $a_2, b_2 \in \mathbb{C}$. Let $a \rightarrow -\infty$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see expression 1 for the \emptyset -Jacobi/Hermite in Table 2)

$$\rho(x, q) = \frac{1}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$$

supported on the set of points $\{\mp q^{\pm k}\alpha\}_{k \in \mathbb{N}_0}$ for arbitrary $\alpha > 0$ (see (2.17) of Theorem 2.3-7).

The OPS in Theorem 5.9 corresponds to the case Ia1 in Chapter 11 and case Va2 in chapter 10 of [19, pages 335, 355-356, 283 and 314-315]. An example of this family is the discrete q -Hermite II polynomials $\tilde{h}_n(x; q)$ satisfying the q -EHT with the coefficients

$$\begin{aligned} \sigma_1(x, q) &= q^{-1}, \quad \sigma_2(x, q) = (x - a_2)(x - b_2), \\ \tau(x, q) &= \frac{1}{q-1}x \quad \text{and} \quad \lambda_n(q) = \frac{1}{1-q}[n]_q \end{aligned}$$

where $a_2 = -i, b_2 = i \in \mathbb{C}$. Discrete q -Hermite II polynomials are orthogonal w.r.t. a measure supported on the set of points $\{\pm q^{\mp k}\}_{k \in \mathbb{N}_0}$ with

$$d_n^2 = (1-q)q^{-n^2}(q; q)_n \frac{(q, -q, -1, -1, -q; q)_\infty}{(i, -i, -iq, iq, -i, i, iq, -iq; q)_\infty}$$

and the conditions $\Lambda_q < 0$ and $a_2, b_2 \in \mathbb{C}$ hold.

5.3 q -Classical \emptyset -Laguerre/Jacobi Polynomials

Let the coefficients σ_2 and σ_1 be linear and quadratic polynomials in x , respectively, such that $\sigma_1(0, q)\sigma_2(0, q) \neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)[x - a_1(q)][x - b_1(q)]$, then from (1.4) $\sigma_2(x, q) = \sigma_2'(0, q)x + \sigma_2(0, q)$ where

$$\sigma_2'(0, q) = -q \left[\frac{1}{2}\sigma_1''(0, q)[a_1(q) + b_1(q)] - (1 - q^{-1})\tau(0, q) \right] \neq 0 \quad \text{and} \quad \sigma_2(0, q) = q \frac{1}{2}\sigma_1''(0, q)a_1(q)b_1(q) \neq 0$$

provided that $\tau'(0, q) = -\frac{\frac{1}{2}\sigma_1''(0, q)}{(1-q^{-1})}$. Therefore, the q -Pearson equation (1.5) takes the form

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{- \left[a_1(q) + b_1(q) - \frac{(1-q^{-1})\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] [x - a_2(q)]}{[qx - a_1(q)][qx - b_1(q)]}$$

where $\left[a_1(q) + b_1(q) - \frac{(1-q^{-1})\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] a_2(q) = a_1(q)b_1(q)$. Let us point out that $f(x, q)$ intersects the y -axis at the point $y = 1$ since $\sigma_2(0, q) = q\sigma_1(0, q)$. On the other hand, we consider the cases depending on the signs of zeros of σ_1 and Λ_q defined by

$$\Lambda_q := \left[a_1(q) + b_1(q) - \frac{(1 - q^{-1})\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right].$$

Case 1. $\Lambda_q < 0$ with $a_1 < 0 < b_1$, **Case 2.** $\Lambda_q > 0$ with $0 < a_1 < b_1$, **Case 3.** $\Lambda_q < 0$ with $0 < a_1 < b_1$.

In Figure 21A, the only possible interval is $(q^{-1}a_1, q^{-1}b_1)$ which is the one described in Theorem 2.3-1. In fact, $\rho(qx, q)/\rho(x, q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$. Then, ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Moreover, $\rho \rightarrow 0$ as $x \rightarrow q^{-1}a_1^+$ and $x \rightarrow q^{-1}b_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. Then there exists an OPS, to be stated in Theorem 5.10, w.r.t. a ρ supported at the points $x = q^k a_1$ and $x = q^k b_1$ for $k \in \mathbb{N}_0$.

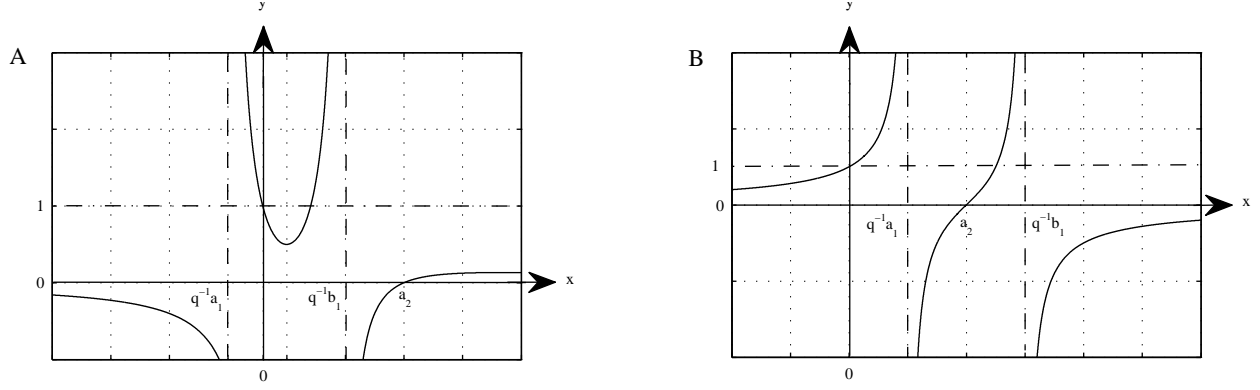


Figure 21: The graph of $f(x, q)$. In A, we have **Case 1** with $\Lambda_q < 0$ and $q^{-1}a_1 < 0 < q^{-1}b_1 < a_2$ and in B, **Case 2** with $\Lambda_q > 0$ and $0 < q^{-1}a_1 < a_2 < q^{-1}b_1$.

Theorem 5.10 *Let $a_1 < 0 < b_1 < a_2$ and $\Lambda_q < 0$. Let $a = a_1$ and $b = b_1$ be the zeros of $\sigma_1(x, q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ w.r.t. weight function (see the 1st expression of the \emptyset -Laguerre/Jacobi case in Table 2)*

$$\rho(x, q) = \frac{(qa^{-1}x, qb^{-1}x; q)_\infty}{(a_2^{-1}x; q)_\infty}$$

supported on $\{q^k a\}_{k \in \mathbb{N}_0} \cup \{q^k b\}_{k \in \mathbb{N}_0}$ (see (2.11) of Theorem 2.3-1).

The OPS in Theorem 5.10 coincides with the case VIIa1 in Chapter 10 of [19, pages 292 and 318]. A typical example of this family is the big q -Laguerre polynomials $P_n(x; a, b; q)$ satisfying the q -EHT with the coefficients

$$\begin{aligned} \sigma_1(x, q) &= q^{-2}(x - a_1)(x - b_1), \quad \sigma_2(x, q) = -abq(x - a_2), \\ \tau(x, q) &= -\frac{q^{-1}}{q-1}x + \frac{a+b-abq}{q-1} \quad \text{and} \quad \lambda_n(q) = \frac{q^{-n}}{q-1}[n]_q \end{aligned}$$

where $a_1 = bq$, $b_1 = aq$ and $a_2 = 1$. The conditions $\Lambda_q < 0$ and $a_1 < 0 < b_1 < a_2$ give the restrictions $b < 0$ and $0 < a < q^{-1}$ on the parameters of $P_n(x; a, b; q)$ with orthogonality on $\{bq, bq^2, bq^3, \dots\} \cup \{a, aq^2, aq^3, \dots\}$ in the sense (2.11) where

$$d_n^2 = (a-b)q(1-q)(-ab)^n q^{n(n+3)/2} (q; q)_n (aq, bq; q)_n \frac{(q, a^{-1}bq, ab^{-1}q; q)_\infty}{(aq, bq; q)_\infty}.$$

In Figure 21B, the only possible interval is $[a_2, q^{-1}b_1)$ which coincides with the one described by Theorem 2.3-3. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $a_2 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$. Thus, ρ is increasing on (a_2, x_0) and decreasing on $(x_0, q^{-1}b_1)$. Moreover, $\rho(qa_2, q) = 0$ and $\rho \rightarrow 0$ as $x \rightarrow q^{-1}b_1^-$ since $\rho(qa_2, q)/\rho(a_2, q) = 0$ and $\rho(qx, q)/\rho(x, q) \rightarrow \infty$ as $x \rightarrow q^{-1}b_1^-$. Therefore, $[a_2, q^{-1}b_1)$ is suitable interval in which we have a positive ρ supported on the set of points $\{q^{-k}a_2\}_{k=0}^N$ where $q^{-N-1}a_2 = q^{-1}b_1$.

Theorem 5.11 *Let $0 < a_1 < a_2 < b_1$ and $\Lambda_q > 0$. Let $a = a_2$ be the zero of $\sigma_2(x, q)$ and $b = q^{-1}b_1$ of $\sigma_1(qx, q)$. Then, there exists a finite family of polynomials $(P_n)_n$ orthogonal w.r.t. the weight function (see the 2nd expression of the \emptyset -Laguerre/Jacobi case in Table 2)*

$$\rho(x, q) = |x|^\alpha \frac{(a/x, qb^{-1}x; q)_\infty}{(a_1(q)/x; q)_\infty}, \quad q^\alpha = -\frac{q^{-2}\sigma_2'(0, q)}{\frac{1}{2}\sigma_1''(0, q)b}$$

supported on $\{q^{-k}a_2\}_{k=0}^N$ where $q^{-N-1}a_2 = q^{-1}b_1$. (see (2.13) of Theorem 2.3-3).

The OPS in Theorem 5.11 coincides with the case IIIb3 in Chapter 11 of [19, pages 343 and 363]. An example of this family is the affine q -Kravchuk polynomials $K_n^{Aff}(x; p, N; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-1}(x - a_1)(x - b_1), \quad \sigma_2(x, q) = -pq^{1-N}(x - a_2),$$

$$\tau(x, q) = \frac{1}{1-q}x - \frac{pq + q^{-N} - pq^{1-N}}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{1}{q-1}[n]_{q^{-1}}$$

where $a_1 = pq$, $b_1 = q^{-N}$ and $a_2 = 1$. The conditions $\Lambda_q > 0$ and $0 < a_1 < a_2 < b_1$ give the constrain $0 < p < q^{-1}$ on the parameter of $K_n^{Aff}(x; p, N; q)$ with orthogonality $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$ in the sense (2.13) where

$$d_n^2 = (-1)^n p^{n-N} (q^{-1} - 1) q^{-N(n+1)} q^{n(n+1)/2} (q, pq, q^{-N}; q)_n \frac{(q, q^{N+1}; q)_\infty}{(pq; q)_\infty}.$$

In the literature, this relation is usually written as a finite sum [19, page 364].

The following four cases listed below fail to define an OPS.

Case 1. $\Lambda_q < 0$ and $q^{-1}a_1 < 0 < a_2 < q^{-1}b_1$, **Case 2.** $\Lambda_q > 0$ and $0 < q^{-1}a_1 < q^{-1}b_1 < a_2$, **Case 2.** $\Lambda_q > 0$ and $0 < a_2 < q^{-1}a_1 < q^{-1}b_1$ and **Case 3.** $\Lambda_q < 0$ and $a_2 < 0 < q^{-1}a_1 < q^{-1}b_1$.

5.4 q -Classical \emptyset -Hermite/Jacobi Polynomials

Let the coefficients σ_2 and σ_1 be constant and quadratic polynomials in x , respectively, such that $\sigma_1(0, q)\sigma_2(0, q) \neq 0$. If σ_1 can be written in terms of its roots, i.e., $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)[x - a_1(q)][x - b_1(q)]$, then, from (1.4)

$$\sigma_2(x, q) = \sigma_2(0, q) = q^{\frac{1}{2}}\sigma_1''(0, q)a_1(q)b_1(q)$$

provided that $(1 - q^{-1})\tau'(0, q) = -\frac{1}{2}\sigma_1''(0, q)$ and $(1 - q^{-1})\tau(0, q) = \frac{1}{2}\sigma_1''(0, q)[a_1(q) + b_1(q)]$. Therefore, the q -Pearson equation (1.5) becomes

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{a_1(q)b_1(q)}{[qx - a_1(q)][qx - b_1(q)]}.$$

Notice that the point $y = 1$ is y -intercept of f . In a similar fashion as before, we introduce the following two cases.

Case1. $a_1(q) < 0 < b_1(q)$. **Case 2.** $0 < a_1(q) < b_1(q)$.

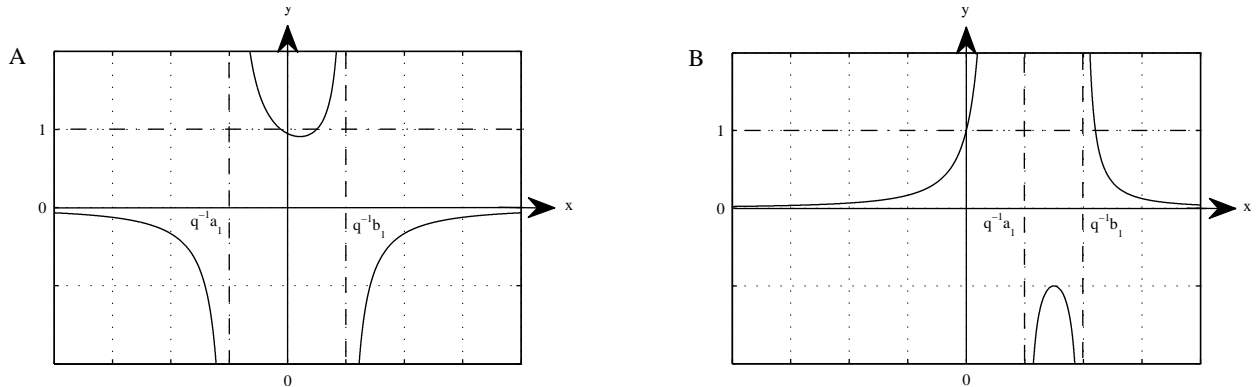


Figure 22: The graph $f(x, q)$ in A, we have Case 1. and in B, Case 2.

In Figure 22A, the only possible interval is $(q^{-1}a_1, q^{-1}b_1)$ which coincides with the Theorem 2.3-1. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}b_1$. Then, ρ is increasing

on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Moreover, $\rho \rightarrow 0$ as $x \rightarrow q^{-1}a_1^+$ and $x \rightarrow q^{-1}b_1^-$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. It is obvious that BC holds at $x = a_1(q)$ and $x = b_1(q)$. Then there exists an OPS with positive q -weight function supported on $\{q^k a_1\}_{k \in \mathbb{N}_0} \cup \{q^k b_1\}_{k \in \mathbb{N}_0}$, as it is stated in the following theorem.

Theorem 5.12 *Let $a_1 < 0 < b_1$. Let $a = a_1$ and $b = b_1$ be the zeros of $\sigma_1(x, q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the \emptyset -Hermite/Jacobi case in Table 2)*

$$\rho(x, q) = (qa^{-1}x, qb^{-1}x; q)_\infty > 0, x \in (a, b)$$

supported on $\{q^k a_1\}_{k \in \mathbb{N}_0} \cup \{q^k b_1\}_{k \in \mathbb{N}_0}$ (see (2.11) of Theorem 2.3-1).

The OPS in Theorem 5.12 coincides with the case VIIa1 in Chapter 10 of [19, pages 292 and 318-320]. An example of this family is Al-Salam-Carlitz I polynomials $U_n^{(a)}(x; q)$ satisfying the q -EHT with the coefficients

$$\begin{aligned} \sigma_1(x, q) &= q^{-1}(x - a_1)(x - b_1), \quad \sigma_2(x, q) = a, \\ \tau(x, q) &= \frac{1}{1-q}x - \frac{1+a}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{q^{1-n}}{q-1}[n]_q \end{aligned}$$

where $a_1 = a$ and $b_1 = 1$. The condition $a_1 < 0 < b_1$ gives the restriction $a < 0$ on the parameter of $U_n^{(a)}(x; q)$ with orthogonality on $\{a, qa, q^2a, \dots\} \cup \{\dots, q^2, q, 1\}$ in the sense (2.11) where

$$d_n^2 = (1-a)(-a)^n q^{\binom{n}{2}} (1-q)(q; q)_n (q, aq, a^{-1}q; q)_\infty.$$

Another example of this family is the discrete q -Hermite I polynomials which are special case of Al-Salam-Carlitz I polynomials (see [19, page 320] for further details). Finally, let us mention that the case represented in Figure 22B is inappropriate to define an OPS.

5.5 q -Classical 0-Jacobi/Laguerre Polynomials

Let σ_2 and σ_1 be quadratic and linear polynomials in x , respectively, such that $\sigma_2(0, q) = \sigma_1(0, q) = 0$. If $\sigma_1(x, q) = \sigma'_1(0, q)x$, then from (1.4), $\sigma_2(x, q) = \frac{1}{2}\sigma''_2(0, q)x^2 + \sigma'_2(0, q)x$ where

$$\frac{1}{2}\sigma''_2(0, q) = q(1-q^{-1})\tau'(0, q) \neq 0 \quad \text{and} \quad \sigma'_2(0, q) = q[\sigma'_1(0, q) + (1-q^{-1})\tau(0, q)] \neq 0$$

provided that $(1-q^{-1})\tau(0, q) \neq -\sigma'_1(0, q)$. For this case the q -Pearson equation reads

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = q^{-1}(1-q^{-1}) \frac{\tau'(0, q)}{\sigma'_1(0, q)} [x - a_2(q)] \quad (5.6)$$

where $-(1-q^{-1})\frac{\tau'(0, q)}{\sigma'_1(0, q)}a_2(q) = 1 + \frac{(1-q^{-1})\tau(0, q)}{\sigma'_1(0, q)}$. Let us point out that f intersects the y -axis at the point

$$y := y_0 = q^{-1} \left[1 + \frac{(1-q^{-1})\tau(0, q)}{\sigma'_1(0, q)} \right].$$

Notice that for the zero cases one of the boundary of (a, b) interval could be zero. This requires to find the behaviour of ρ at the origin.

Lemma 5.13 *If $0 < y_0 < 1$, then $\rho(z, q) \rightarrow 0$ as $z \rightarrow 0$. Otherwise it diverges to $\mp\infty$.*

Proof: From (5.6) it follows that

$$\rho(q^k x, q) = q^{-k} \left[1 + \frac{(1 - q^{-1})\tau(0, q)}{\sigma'_1(0, q)} \right]^k (x/a_2(q); q)_k \rho(x, q)$$

from where the result follows. ■

Again we identify the cases depending on σ_2 , $\Lambda_q := \frac{\tau'(0, q)}{\sigma'_1(0, q)}$ and y_0 .

Case 1. $\Lambda_q > 0$, $a_2 > 0$ and $y_0 > 1$, **Case 2.** $\Lambda_q < 0$, $a_2 < 0$ and $0 < y_0 < 1$, **Case 3.** $\Lambda_q < 0$, $a_2 > 0$ and $y_0 < 0$.

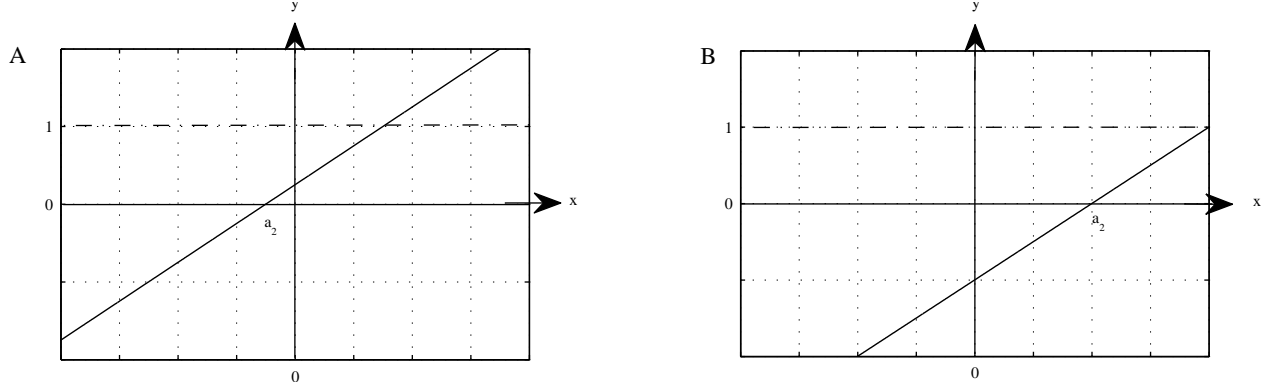


Figure 23: The graph of $f(x, q)$ in A, we have Case 2. and in B, Case 3.

The **Case 1**, do not lead to any OPS. **Case 2-3** are introduced in Figure 23. In Figure 23A, the only possible interval is $(0, \infty)$ which coincides with 6th case of Theorem 2.3. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > 0$. Then ρ is increasing on $(0, x_0)$ and decreasing on (x_0, ∞) . Furthermore, $\rho \rightarrow 0$ as $x \rightarrow 0^+$ by Lemma 5.13 since $0 < y_0 < 1$ and $\rho \rightarrow 0$ as $x \rightarrow \infty$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. Therefore, it could be possible to have a suitable ρ on $(0, \infty)$. But we need to check $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ for $k \in \mathbb{N}_0$ by using extended q -Pearson equation (4.3). It is clear from (4.3) that graph of the function g defined in (4.3) looks like the one represented in Figure 23A with y -intercept, $0 < q^{k+1}y_0 < 1$, $k \in \mathbb{N}_0$. Thus $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ for $k \in \mathbb{N}_0$ and therefore, there exists an OPS supported on $\{q^{\pm k}\}_{k \in \mathbb{N}_0}$ which is established in the next theorem.

Theorem 5.14 *Let $\Lambda_q < 0$, $a_2 < 0$ and $0 < qy_0 < 1$. Let $a = 0$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 1st expression of the 0-Jacobi/Laguerre case in Table 2)*

$$\rho(x, q) = |x|^\alpha \frac{1}{(a_2^{-1}x; q)_\infty}, \quad q^\alpha = -\frac{q^{-2\frac{1}{2}}\sigma_2''(0, q)a_2}{\sigma'_1(0, q)}$$

supported on $\{q^{\pm k}\}_{k \in \mathbb{N}_0}$ (see (2.16) of Theorem 2.3-6).

The OPS in Theorem 5.14 coincides with the case IIIa2 in Chapter 10 of [19, pages 272 and 309]. An example of this family is the q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$ satisfying the q -EHT with the coefficients

$$\begin{aligned} \sigma_1(x, q) &= q^{-2}x, \quad \sigma_2(x, q) = q^\alpha x(x - a_2), \\ \tau(x, q) &= -\frac{q^\alpha}{1 - q}x + \frac{q^{-1} - q^\alpha}{1 - q} \quad \text{and} \quad \lambda_n(q) = [n]_q \frac{q^\alpha}{1 - q} \end{aligned}$$

where $a_2 = -1$. The conditions $\Lambda_q < 0$, $a_2 < 0$ and $0 < qq_0 < 1$ give the constrain $\alpha > -1$ on the parameter of $L_n^{(\alpha)}(x; q)$ with orthogonality on $\{q^{\pm k}\}_{k \in \mathbb{N}_0}$ in the sense (2.16) where

$$d_n^2 = q^{-n}(1-q) \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \frac{(q, -q^{\alpha+1}, -q^{-\alpha}; q)_\infty}{(q^{\alpha+1}, -q, -q; q)_\infty}.$$

In Figure 23B, the positivity of ρ enables us to skip the intervals $(-\infty, 0)$ and $(0, a_2)$. So the only interval is (a_2, ∞) which is the one described in Theorem 2.3-5. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > a_2$. Therefore, ρ is increasing on (a_2, x_0) and decreasing on (x_0, ∞) . Moreover, $\rho(qa_2, q) = 0$ since $\rho(qa_2, q)/\rho(a_2, q) = 0$ and $\rho \rightarrow 0$ as $x \rightarrow \infty$ since $\rho(qx, q)/\rho(x, q) \rightarrow \infty$. Furthermore, since the graph of the function g defined in (4.3) looks like the one represented in Figure 23B one can conclude that $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow \infty$ for $k \in \mathbb{N}_0$ and therefore we have the following theorem.

Theorem 5.15 *Let $\Lambda_q < 0$, $a_2 > 0$ and $qq_0 < 0$. Let $a = a_2$ be the zero of $\sigma_2(x, q)$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 2nd expression of the 0-Jacobi/Laguerre case in Table 2)*

$$\rho(x, q) = |x|^\alpha \sqrt{x^{\log_q x - 1}} (qa/x; q)_\infty, \quad q^\alpha = \frac{q^{-2\frac{1}{2}} \sigma_2''(0, q)}{\sigma_1'(0, q)}$$

supported on the set of points $\{q^{-k}a_2\}_{k \in \mathbb{N}_0}$ (see (2.15) of Theorem 2.3-5).

The OPS in Theorem 5.15 coincides with the case IIa2 in Chapter 11 of [19, pages 337 and 358]. An example of this family is the q -Charlier polynomials $C_n(x; a; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = aq^{-2}x, \quad \sigma_2(x, q) = x(x - a_2),$$

$$\tau(x, q) = -\frac{1}{1-q}x + \frac{a+q}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = [n]_q \frac{1}{1-q}$$

where $a_2 = 1$. The conditions $\Lambda_q < 0$, $a_2 > 0$ and $qq_0 < 0$ give the restriction $a > 0$ on the parameter of $C_n(x; a; q)$ with orthogonality on $\{1, q^{-1}, q^{-2}, \dots\}$ in the sense (2.15) where

$$d_n^2 = a^{2n} q^{-n(2n+1)} (-a^{-1}q, q; q)_n (-a, q; q)_\infty.$$

In the literature, this relation is usually written as an infinite sum [19, page 360].

5.6 q -Classical 0-Bessel/Jacobi Polynomials

Let σ_2 and σ_1 be quadratic polynomials in x , respectively, such that $\sigma_2'(0, q) = 0$ and $\sigma_2(0, q) = \sigma_1(0, q) = 0$. If $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)x[x - a_1(q)]$, $\frac{\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \neq -\frac{1}{(1-q^{-1})}$ and $\frac{\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} = \frac{a_1(q)}{(1-q^{-1})}$, then from (1.4) we have $\sigma_2(x, q) = \frac{1}{2}\sigma_2''(0, q)x^2 = q \left[\frac{1}{2}\sigma_1''(0, q) + (1-q^{-1})\tau'(0, q) \right] x^2$. As a result, the q -Pearson equation (1.5) becomes

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{\left[1 + \frac{(1-q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] x}{q[qx - a_1(q)]}.$$

Let us point out that $f(x, q)$ passes through the origin and the line $y = \Lambda_q := q^{-2} \left[1 + \frac{(1-q^{-1})\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right] \neq 0$ is its horizontal asymptote. Hence, we have the following two cases:

Case 1. $\Lambda_q < 0$ and $a_1 > 0$ and **Case 2.** $\Lambda_q > 0$ and $a_1 > 0$.

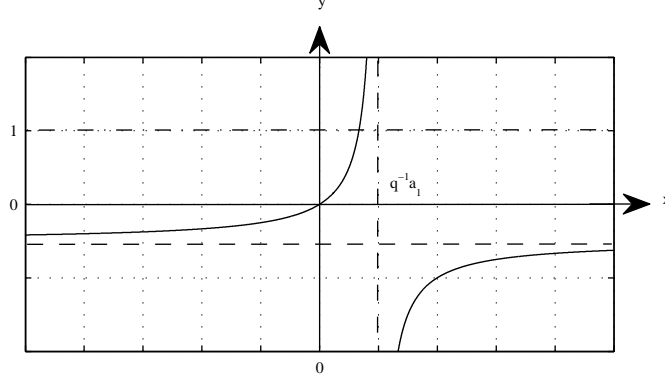


Figure 24: The graph of $f(x, q)$ in Case 1.

The **Case 2** with $\Lambda_q > 1$ and $0 < \Lambda_q < 1$ do not lead to any OPS. The **Case 1** is represented in Figure 24 from where it follows that the only possible interval is $(0, q^{-1}a_1)$ which is the one defined in Theorem 2.3-2. Notice also that $\rho(qx, q)/(x, q) = 1$ at $0 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}a_1$. Then, ρ is increasing on $(0, x_0)$ and decreasing on $(x_0, q^{-1}a_1)$. Moreover, $\rho \rightarrow 0$ as $x \rightarrow 0^+$ and $x \rightarrow q^{-1}a_1^-$ since $\rho(qx, q)/(x, q) \rightarrow 0$ and $\rho(qx, q)/(x, q) \rightarrow \infty$, respectively. Then, there exists an OPS with a suitable ρ defined on $(0, a_1]$ supported at the points $a_1 q^k$ for $k \in \mathbb{N}_0$ and the following theorem holds.

Theorem 5.16 *Let $q^2\Lambda_q < 0$ and $a_1 > 0$. Let $a = 0$ and $b = a_1$ be the zeros of $\sigma_1(x, q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 0-Bessel/Jacobi case in Table 2)*

$$\rho(x, q) = |x|^\alpha \sqrt{x^{\log_q x - 1}} (b^{-1}qx; q)_\infty, \quad q^\alpha = -\frac{q^{-2\frac{1}{2}}\sigma_2''(0, q)}{\frac{1}{2}\sigma_1''(0, q)b}$$

supported on the set of points $\{q^k a_1\}_{k \in \mathbb{N}_0}$ (see (2.12) of Theorem 2.3-2).

The OPS in Theorem 5.16 coincides with the case IVa5 in Chapter 10 of [19, pages 278 and 313]. An example of this family is the Alternative q -Charlier (q -Bessel) polynomials $K_n(x; a; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = -q^{-2}x(x - a_1), \quad \sigma_2(x, q) = ax^2, \\ \tau(x, q) = -\frac{1 + aq}{(1 - q)q}x + \frac{1}{(1 - q)q} \quad \text{and} \quad \lambda_n(q) = q^{-n}[n]_q \frac{1 + aq^n}{1 - q}$$

where $a_1 = 1$. The conditions $q^2\Lambda_q < 0$ and $a_1 > 0$ give the constrain $a > 0$ on the parameter of $K_n(x; a; q)$ with orthogonality on $\{\dots, q^2, q, 1\}$ in the sense (2.12) where

$$d_n^2 = a^n q^{n(3n-1)/2} (-aq, q; q)_\infty \frac{(q, -a; q)_n}{(-a, -aq; q)_{2n}}.$$

In the literature, this relation can be found as an infinite sum [19, page 314].

5.7 q -Classical 0-Bessel/Laguerre Polynomials

Let σ_2 and σ_1 be quadratic and linear polynomials in x , respectively, such that $\sigma_2'(0, q) = 0$ and $\sigma_2(0, q) = \sigma_1(0, q) = 0$. If $\sigma_1(x, q) = \sigma_1'(0, q)x$, then, from (1.4) $\sigma_2(x, q) = \frac{1}{2}\sigma_2''(0, q)x^2 = q(1 - q^{-1})\tau'(0, q)x^2$ provided that $(1 - q^{-1})\tau(0, q) = -\sigma_1'(0, q)$. So the q -Pearson equation is now

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = q^{-1}(1 - q^{-1}) \frac{\tau'(0, q)}{\sigma_1'(0, q)} x.$$

Clearly, f passes through the origin. According to the sign of $\Lambda_q := \frac{\tau'(0,q)}{\sigma_1'(0,q)}$ we have only one possible case.

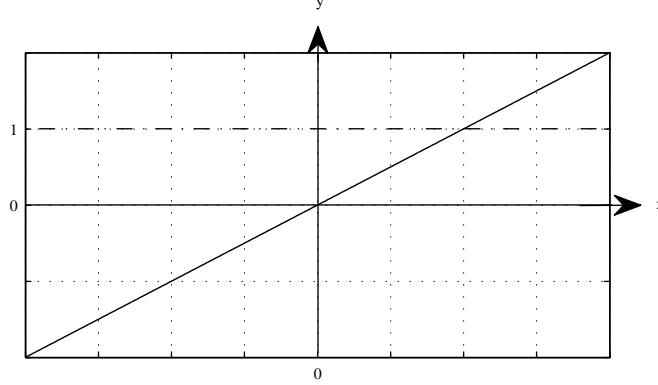


Figure 25: The graph of $f(x, q)$ with $\Lambda_q < 0$, $a_2 = 0$.

From Figure 25 it follows that $(0, \infty)$ is the only possible interval and it coincides with the one described in Theorem 2.3-6. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) > 0$. Then, ρ is increasing on $(0, x_0)$ and decreasing on (x_0, ∞) . Moreover, by use of the extended q -Pearson equation (4.3) it is straightforward to see that $\sigma_1(x, q)\rho(x, q)x^k \rightarrow 0$ as $x \rightarrow +\infty$. Thus, the following theorem holds.

Theorem 5.17 *Let $\Lambda_q < 0$, $a_2 = 0$ and $qy_0 = 0$. Let $a = 0$ and $b \rightarrow \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 0-Bessel/Laguerre case in Table 2)*

$$\rho(x, q) = |x|^\alpha \sqrt{x^{\log_q x - 1}}, \quad q^\alpha = \frac{q^{-2} \frac{1}{2} \sigma_2''(0, q)}{\sigma_1'(0, q)}$$

supported on $\{q^{\mp k}\}_{k \in \mathbb{N}_0}$ (see (2.16) of Theorem 2.3-6).

The OPS in Theorem 5.17 coincides with the case IIIa2 in Chapter 10 of [19, pages 272 and 309]. An example of this family is Stieltjes-Wigert polynomials $S_n(x; q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-2}x, \quad \sigma_2(x, q) = x^2, \\ \tau(x, q) = -\frac{1}{1-q}x + \frac{1}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = [n]_q \frac{1}{1-q}.$$

The conditions $\Lambda_q < 0$, $a_2 = 0$ and $qy_0 = 0$ are satisfied for $S_n(x; q)$ and they are orthogonal w.r.t. a measure supported on $\{q^{\mp k}\}_{k \in \mathbb{N}_0}$ in the sense (2.16) with

$$d_n^2 = q^{-n}(1-q) \frac{(-tq, -1/t, q; q)_\infty}{(q^2; q)_n}.$$

5.8 q -Classical 0-Laguerre/Jacobi Polynomials

Let σ_2 and σ_1 be linear and quadratic polynomials in x , respectively, such that $\sigma_2(0, q) = \sigma_1(0, q) = 0$. If $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)x[x - a_1(q)]$ and $\frac{\tau'(0, q)}{\frac{1}{2}\sigma_1''(0, q)} = -\frac{1}{(1-q^{-1})}$, then from (1.4) we get $\sigma_2(x, q) = \sigma_2'(0, q)x = q[(1 - q^{-1})\tau(0, q) - \frac{1}{2}\sigma_1''(0, q)a_1(q)]x$. Therefore, the q -Pearson equation has the form

$$f(x, q) := \frac{\rho(qx, q)}{\rho(x, q)} = \frac{(1 - q^{-1})\frac{\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} - a_1(q)}{q[qx - a_1(q)]}.$$

Notice that $y = 0$ is the horizontal asymptote of $f(x, q)$, and its y -intercept is

$$y := y_0 = q^{-1} \left[1 - \frac{(1 - q^{-1})}{a_1(q)} \frac{\tau(0, q)}{\frac{1}{2}\sigma_1''(0, q)} \right].$$

We have the following two cases: **Case 1.** $y_0 > 0$ and $a_1 > 0$, **Case 2.** $y_0 < 0$ and $a_1 > 0$.

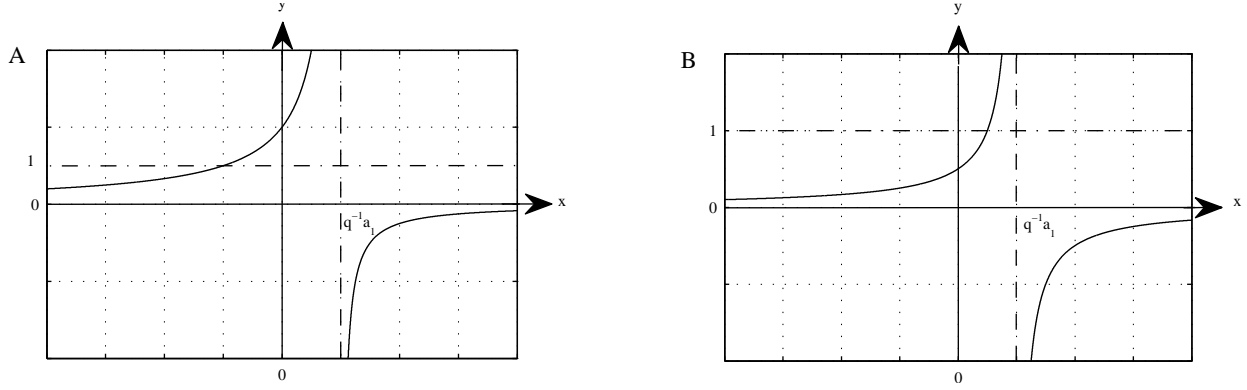


Figure 26: The graph of $f(x, q)$ in **Case 1**. In A, we have $y_0 > 1$ and $a_1 > 0$ and in B, $0 < y_0 < 1$ and $a_1 > 0$.

The Case 1 represented in Figure 26A as well as the **Case 2** do not yield any OPS. From Figure 26B, it follows that the only possible interval is $(0, q^{-1}a_1)$ which coincides with the 2nd case of Theorem 2.3. A completely similar analysis as the one done in the previous case allows us to conclude that in $(0, a_1]$ an OPS can be defined which is orthogonal w.r.t. a suitable ρ supported on the set of points $\{q^k a_1\}_{k \in \mathbb{N}_0}$. Then, we have the following Theorem.

Theorem 5.18 *Let $a_1 > 0$ and $0 < qy_0 < 1$. Let $a = 0$ and $b = a_1$ be the zeros of $\sigma_1(x, q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 0-Laguerre/Jacobi case in Table 2)*

$$\rho(x, q) = |x|^\alpha (b^{-1}qx; q)_\infty, \quad q^\alpha = -\frac{q^{-2\frac{1}{2}}\sigma_2''(0, q)}{\frac{1}{2}\sigma_1''(0, q)b}$$

supported on the set of points $\{q^k a_1\}_{k \in \mathbb{N}_0}$. (see (2.12) of Theorem 2.3-2).

The OPS in Theorem 5.18 coincides with the case IVa4 in Chapter 10 of [19, pages 278 and 312]. An example of this family is the little q -Laguerre (Wall) polynomials $P_n(x; \alpha|q)$ satisfying the q -EHT with the coefficients

$$\sigma_1(x, q) = q^{-2}x(a_1 - x), \quad \sigma_2(x, q) = ax,$$

$$\tau(x, q) = -\frac{1}{(1-q)q}x + \frac{1-aq}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = \frac{q^{-n}}{1-q}[n]_q$$

where $a_1 = 1$. The conditions $0 < qy_0 < 1$ and $a_1 > 0$ give the restriction $0 < a < q^{-1}$ on the parameter of $P_n(x; \alpha|q)$ with orthogonality on $\{\dots, q^2, q, 1\}$ in the sense (2.12) where

$$d_n^2 = a^n q^{n^2} \frac{(q; q)_\infty}{(aq; q)_\infty} (q, aq; q)_n.$$

In the literature, this relation can be found as an infinite sum [19, page 312].

6 Concluding remarks

The q -polynomials of the Hahn class have been revisited by use of a direct and very simple geometrical approach based on the qualitative analysis of solutions of the q -Pearson (1.5) and the *extended* q -Pearson (4.3) equations. By this way, it is shown that it is possible to introduce in a unified manner all orthogonal polynomial solutions of the q -EHT, which are orthogonal w.r.t. a measure supported on some set of points in certain intervals. In this review article we are able to extend the well known orthogonality relations for the big q -Jacobi polynomials (see Theorem 4.5 and Theorem 4.7), q -Hahn polynomials (see Theorem 4.6 and Theorem 4.8), and for the q -Meixner polynomials (see Theorem 5.4) to a larger set of their parameters.

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